

Lec27-simplex-method

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Linear programming

$$\begin{aligned} & \text{maximize } \vec{c} \cdot \vec{x} \\ & \text{subject to } A\vec{x} \leq \vec{b} \\ & \text{and } \vec{x} \geq 0 \end{aligned}$$

$\vec{c} \in \mathbb{R}^n$
 $\vec{x} \in \mathbb{R}^n$
 $A \in \mathbb{R}^{m \times n}$
 $\vec{b} \in \mathbb{R}^m$

Minimization, " \geq " + " $=$ "
can be reduced to this

standard form when we
add constraint that all
variables are positive

Notice: given a problem with potentially negative variables, we
can replace with two positive variables.

$$\begin{aligned} \text{constraints } \begin{cases} 2x_1 + x_2 \leq 10 \\ -x_1 \leq -1 \end{cases} & \Rightarrow \begin{cases} 2x_1 + x_2^+ - x_2^- \leq 10 \\ -x_1 \leq -1 \end{cases}, \text{ where } x_2^+, x_2^- \geq 0. \\ \text{obj: } x_1 + x_2 & \quad \text{obj: } x_1 + x_2^+ - x_2^- \end{aligned}$$

x_1 already positive
 $x_2^+ - x_2^- = x_2 \in \mathbb{R}$
 any pos or neg number

Thus, the above standard form works for all LP.

Idea behind simplex: Increase objective by repeatedly turning up a
variable until it can no longer be increased.

Example simplex run

All $s_i \geq 0$, slack variables

$$\text{Constraints } \begin{cases} x_1 + x_2 \leq 3 \\ -x_1 + 3x_2 \leq 1 \\ x_2 \leq 3 \end{cases}$$

$$\begin{aligned} x_1 + x_2 + s_1 &= 3 \\ -x_1 + 3x_2 + s_2 &= 1 \\ x_2 + s_3 &= 3 \end{aligned}$$

easy to find a solution
to constraints, but equivalent
to optimize.

$$\text{Maximize } x_1 + x_2 = z$$

$$\text{Maximize } x_1 + x_2 = z$$

Set $s_1 = 3$ $x_1 = 0$
 $s_2 = 1$ $x_2 = 0$
 $s_3 = 3$

Also, all $x_i \geq 0$.

Tableau
(RHS vars = 0)
LHS may be nonzero

(harder when RHS is negative)
will see later

$$s_1 = 3 - x_1 - x_2$$

$$s_2 = 1 + x_1 - 3x_2$$

$$s_3 = 3 - x_2$$

$$z = 0 + x_1 + x_2$$

To increase z , we want to increase
a variable with pos. coefficient

Pick x_1 as "entering variable"

$$\begin{cases} s_1 = 3 - x_2 \geq 0 \\ x_2 \leq 3 \end{cases}$$

... 1

$t = 0$ a variable with pos. coefficient

Pick x_2 as "entering variable"
 Variables on LHS must also change in value,
 but can't be negative

$$\left\{ \begin{array}{l} s_1 = 3 - x_2 \geq 0 \\ s_2 = 1 - 3x_2 \geq 0 \\ s_3 = 3 - x_2 \geq 0 \end{array} \right. \left\{ \begin{array}{l} x_2 \leq 3 \\ x_2 \leq \frac{1}{3} \\ x_2 \leq 3 \end{array} \right. \leftarrow \text{strictest inequality}$$

choose s_2 as "leaving variable"

set $x_2 = \frac{1}{3}$

$$s_2 = 1 + x_1 - 3x_2$$

$$x_2 = \frac{1}{3} + \frac{1}{3}x_1 - \frac{1}{3}s_2$$

rewrite tableau with x_2 on LHS, s_2 on RHS

$$s_1 = 3 - x_1 - \left(\frac{1}{3} + \frac{1}{3}x_1 - \frac{1}{3}s_2\right) = \frac{8}{3} - \frac{4}{3}x_1 + \frac{1}{3}s_2$$

$$x_2 = \frac{1}{3} + \frac{1}{3}x_1 - \frac{1}{3}s_2$$

$$s_3 = 3 - \left(\frac{1}{3} + \frac{1}{3}x_1 - \frac{1}{3}s_2\right) = \frac{8}{3} - \frac{1}{3}x_1 + \frac{1}{3}s_2$$

$$z = x_1 + \left(\frac{1}{3} + \frac{1}{3}x_1 - \frac{1}{3}s_2\right) = \frac{1}{3} + \frac{4}{3}x_1 - \frac{1}{3}s_2$$

objective now $\frac{1}{3}$

to increase z ,
 increase a variable
 with positive coefficients

Choose x_1 as
 leaving var.

Find strictest
 constraint:

$$\left\{ \begin{array}{l} \frac{8}{3} - \frac{4}{3}x_1 \geq 0 \\ \frac{1}{3} + \frac{1}{3}x_1 \geq 0 \\ \frac{8}{3} - \frac{1}{3}x_1 \geq 0 \end{array} \right. \left\{ \begin{array}{l} x_1 \leq 2 \\ x_1 \geq -1 \\ x_1 \leq 8 \end{array} \right. \leftarrow \text{strictest constraint}$$

Not a constraint at all because x_1 had pos. coeff.

$$s_1 = \frac{8}{3} - \frac{4}{3}x_1 + \frac{1}{3}s_2 \Rightarrow x_1 = 2 - \frac{3}{4}s_1 + \frac{1}{4}s_2$$

Substitute in x_1 to all equations

$$x_1 = 2 - \frac{3}{4}s_1 + \frac{1}{4}s_2$$

$$x_2 = 1 - \frac{1}{4}s_1 - \frac{1}{4}s_2$$

$$s_3 = 2 + \frac{1}{4}s_1 + \frac{1}{4}s_2$$

$$z = 3 - s_1$$

Note: increasing
 x_1 also increased
 x_2

done since all vars have neg coeff.

Optimal value is $z=3$ with $x_2=2$, $x_1=1$.

Can check that this satisfies original constraints.

Summary of simplex so far

1. Write LP with slack variables, & get initial sol.
2. Choose variable v in obj with pos coeff. to increase
3. Choose the strictest constraint (will be one where v has neg. coeff.)
4. Rewrite strictest eq with v on LHS, & substitute for v everywhere else.
5. If all coeff in obj are neg. done. Else, go back to step 2.

pivot: variables on LHS are "basic" variables.
each pivot swaps a different variable in as a basic var while improving (or at least not decreasing) obj.
(may change value of other basic vars)

If no constraint provides upper bound, then problem is unbounded,

If strictest upper bound is 0, then \exists other feasible sol of equiv. cost.
 \Rightarrow may have to pivot w/o increasing obj to make progress.
(or decreasing)

Pivot rules:

- largest coeff in obj
- largest increase after pivot in obj
- random

\leftarrow max increase per unit of entering var when ignoring other vars

• steepest edge $\frac{\partial \cdot (\vec{x}_{\text{new}} - \vec{x}_{\text{old}})}{\|\vec{x}_{\text{new}} - \vec{x}_{\text{old}}\|}$ } max increase per unit of change in overall \vec{x} vector

- Bland's rule - choose entering var w/ lowest index (and leaving var with lowest index)

\hookrightarrow prevents infinite cycling. Not practical, but important theoretically.
almost never happens even w/ other rules.
... instead like it is w/ Bland.

→ prevents infinite cycling.
 almost never happens even w/ other rules,
 but not guaranteed like it is w/ Bland.

Optimality

The final objective is equivalent in value to original.

And is always (e.g. $z = 24 - 5x_1 - 3x_2$) with only neg coeff., so we can't do better by any more pivots.

Initialization

We assumed above that $b_i \geq 0$, so we could set non-slack vars to 0 and slack vars to b_i , for an initial feasible sol.

What if $b_i < 0$?

Ex

$$\begin{array}{r} s_1 = 3 - x_1 - x_2 \\ s_2 = -1 + x_1 - 3x_2 \\ s_3 = 3 - x_2 \end{array}$$

$$z = 0 + x_1 + x_2$$

← can't set $s_2 = -1$.

Auxiliary problem

Given LP $\max \vec{c} \cdot \vec{x}$ s.t. $A\vec{x} \leq \vec{b}$, $\vec{x} \geq 0$

Auxiliary LP $\max -y$ s.t. $A\vec{x} - y \leq \vec{b}$, $\vec{x} \geq 0$, $y \geq 0$

← ignore original opt

new var y

Introduce slack variables $s_i \geq 0$ as usual, but for the initial feasible sol, set $x_i = 0$, $y = \min_i b_i$, and $s_i = b_i - y$, all non-negative.

In auxiliary LP, we are trying to get rid of y .

L if we can't, then no feasible sol

L if we do, then we have feasible sol to original LP, and

- continue the tableau from there, dropping all mention of y (which is 0)

L iff we do, then we have feasible sol to original LP, and
can continue the tableau from there, dropping all mention of γ (which is 0)