Lec28-NP-hardness

KT 8.1, 8.2, 8.3, 8.4

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Polynomial time: e.g. $O(m^2 n)$

Often we want poly-time because exp algs $O(2^n)$ are too slow.

Mistake: NP-hard $\neq$ non-polynomial time

not the correct definition

Decision vs. Optimization problems

Optimization: output answer that minimizes / maximizes an objective function

Ex. All of linear programming.

Shortest path

Minimum spanning tree.

Max-Flow

Decision: Yes/No answer to whether there exists a solution that is good enough

Ex. $\exists$ s-t path shorter than 5

$\exists$ MST with weight ≤ 100

Ex. Circulation with demands

Optimization → Decision

Problem (MST) Given a graph $G=(V, E)$, with weights w_e on each edge in E, find a spanning tree of smallest total weight.

Problem (MST-Decision) Given a number C, and a graph $G=(V, E)$, with weights w_e on each edge in E, is there a spanning tree of total weight less than C

Glossary

Problem: a decision problem in general terms

Instance: A particular set of input into a problem:

e.g. $C = 5$ $G =$ 

Yes-Instance: An instance with answer "YES"

No-Instance: An instance with answer "NO"

$C = 2$ $G =$ 

deciding: determining if an Instance has answer "YES" or "NO"

Define: The class P = set of problems.

A problem $Q \in P$ if

$\exists$ algorithm A_Q s.t. $\forall$ instances $I \in Q$, on a deterministic Turing machine

$A_Q(I)$ runs in fewer than poly$(|I|)$ steps, and s.t.

if I is a YES-instance, $A_Q(I) = YES$

if I is a NO-instance, $A_Q(I) = NO$.

P is the set of problems that can be decided in polynomial number of steps. i.e. "efficient" solutions.

Define: The class NP = set of problems.

A problem $Q \in NP$ if a non-deterministic Turing machine can return YES on a YES instance in poly-time.

Easier Definition: A problem $Q \in NP$ if certificate evidence of a YES-instance can be verified in poly-time. (on normal Turing machine)

Ex. (Traveling Salesman - Decision)

Given distances d_{ij} between n cities, and a number C,

Ex. [illegible]

Given distances d_{ij} between n cities, and a number C, is there a way to visit each city exactly once where the total length of the route is $\leq C$.

If someone gives you a route, it is easy to check if it provides a solution that makes this a YES-instance.

Ex. (Independent Set - Decision)

Given graph G, $\exists\ S \subseteq V$, $|S| \geq k$ s.t. no two nodes in S are connected by an edge?

Finding set is hard, (later)

Checking a set S^* I give you is easy.

S^* acts as a certificate that $\langle G, k \rangle$ is a Yes-instance

Define: NP = set of problems Q for which

$\exists$ algorithm C_Q s.t.

$\forall$ instances $I \in Q$,

if I is yes-instance, $C_Q(I, S) = YES$ for some S

if I is no-instance, $C_Q(I, S) = NO$ $\forall S$.

and $C_Q(I, S)$ runs in poly($|I|$) time.

$C_Q(I, S)$: verifier (efficient certifier)

S: certificate

(nondeterministic polynomial)

(nondeterministic polynomial)

NP – set of problems with efficient certifier

P – set of problems with efficient certifier
(polynomial) that ignores the certificate

Thm $P \subseteq NP$.

proof. Suppose $X \in P$. Then $\exists$ poly-time alg A that decides X.

To show $X \in NP$, need to design efficient certifier $C_X(I,S)$.

Just take $C_X(I,S) = A(I)$. □

every problem with poly-time alg is in NP.

Question: $P \stackrel{?}{=} NP$

Know $P \subseteq NP$.

Does there $\exists$ a problem in NP that is NOT in P?

Is checking a solution actually easier than finding a solution? Intuitively maybe.

No one knows answer, but most people believe it to be true.

Reductions as tool for hardness

We want to say some problems are hard.

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Easier 1st step: Problem X is at least as hard as Problem Y.

Proof technique: we reduce Problem Y to Problem X:

Suppose we have black box that solves instances of X.

How can you solves instances of Y using poly steps + poly calls to black box.

If Y can be reduced to X, $Y \leq_p X$

Y is poly-time reducible to X

$\Rightarrow$ X is at least as hard as Y because if you can solve X, you can solve Y.

Note: reduce to problem we want to show is harder

Examples:

Max Bipartite Matching $\leq_p$ Max Network Flow

Image Segmentation $\leq_p$ Min-Cut

Max Network Flow $\leq_p$ Linear Programming

Circulation w/ Demands & Lower Bounds $\leq_p$ Circulation w/ Demands $\leq_p$ Max Flow

If ILP can be reduced to TSP, $ILP \leq_p TSP$

$\Rightarrow$ if we can solve TSP, we can solve ILP.

If ILP can be [illegible]

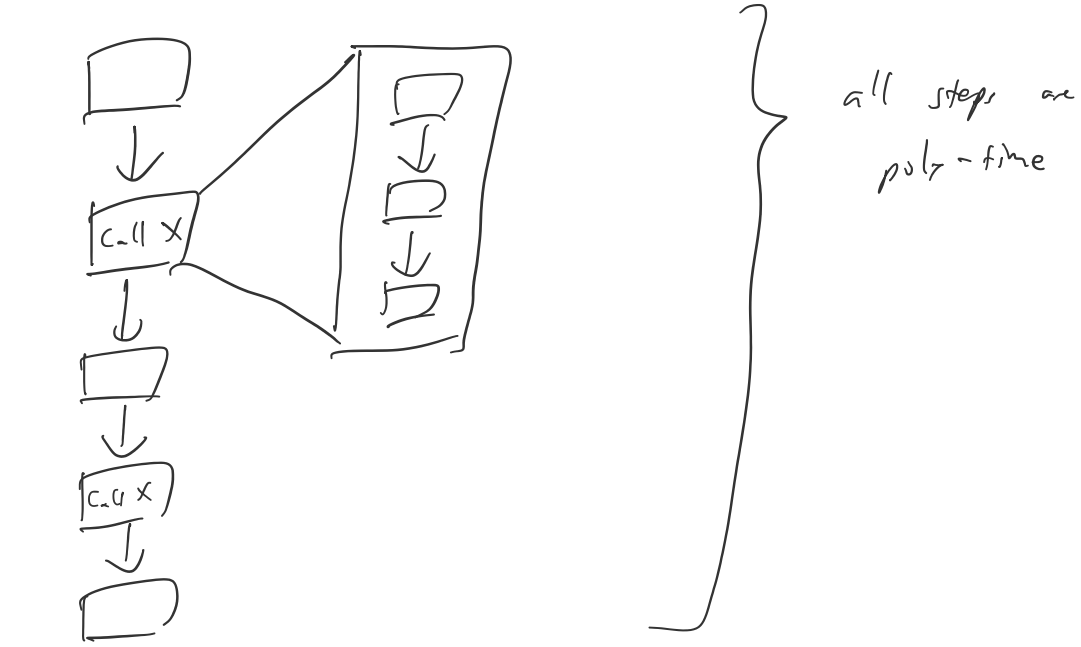
⇒ if we can solve TSP, we can solve ILP.

Suppose $Y \leq_p X$ and $\exists$ poly-time alg for X.

Then $\exists$ poly-time alg for Y.

Why? Polynomials compose

Alg for Y



Thm: If $Y \leq_p X$, and Y CANNOT be solved in poly-time, then X cannot be solved in poly-time.

proof. Suppose we have alg. A that solves X in poly-time,

Then we reduce Y to X, giving a solution to Y in poly time a contradiction

So if we can find a hard problem Y, can show

So if we can find a hard problem Y, can show a bunch of other problems are hard by reducing Y to those other problems.

Define: A problem X is NP-hard if $\forall\, Y \in NP$, $Y \leq_p X$.

In other words, these are problems that are at least as hard as every problem in NP.

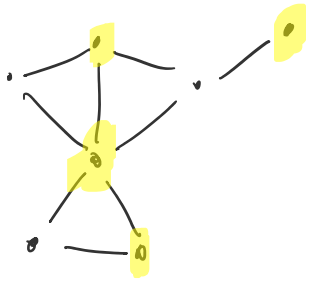
Lemma: If X is NP-hard, and $X \leq_p Z$, then Z is NP-hard.

proof. If we can reduce any $Y \in NP$ to X, and we can reduce X to Z, then clearly, we can reduce Y to Z.

Vertex Cover

Def. A vertex cover of graph $G=(V,E)$ is a set $S \subseteq V$ s.t. $\forall\, e \in E$, at least one endpoint of e is in S.

Ex



Fact: Vertex Cover is NP-hard.

We can now use vertex cover to prove other problems NP-hard.

Independent Set $\leq_p$ Vertex Cover

Recall: An independent set in a graph $G=(V,E)$ is a

Recall: An independent set in a graph $G=(V,E)$ is a set $S \subseteq V$ s.t. $\forall\, x,y \in S$, $(x,y) \notin E$.

Thm If $G=(V,E)$ is a graph, then S being an ind. set $\iff$ $V-S$ is a vertex cover.

proof. ($\Rightarrow$) Suppose S is an independent set, and let $e=(u,v) \in E$.
Only one of u, v can be in S.
$\Rightarrow$ At least one of u, v is in $V-S$.
$\Rightarrow$ $V-S$ is a vertex cover.

($\Leftarrow$) Suppose $V-S$ is a vertex cover, and let $u, v \in S$.
$\Rightarrow$ $(u,v) \notin E$, as that edge isn't covered by $V-S$.
$\Rightarrow$ S is an ind. set. □

So vertex covers and independent sets are complements.

Given an instance of Independent Set $\langle G, k \rangle$,
we ask our Vertex Cover black box if $\exists$ a vertex cover C of size $\leq |V| - k$.
If so, we define $S = V - C$ as our sol to Independent Set, as S is an ind set of size k.

If not, then no vertex cover of size $\leq |V| - k$, so no independent set of size $\geq k$.

$\Rightarrow$ Ind Set $\leq_p$ Vertex Cover.

Claim: Vertex Cover $\leq_p$ Ind Set.

proof To decide if G has a vertex cover of size k,

Claim:

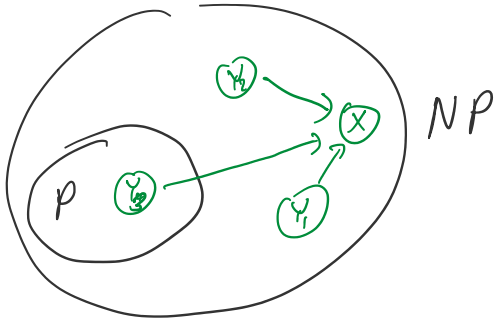
proof. To decide if G has a vertex cover of size k, we ask if it has an ind. set of size $|V| - k$.

$\Rightarrow$ Vertex Cover & Ind Set are equivalently difficult.

Define: X is NP-complete if

- $X \in NP$
- X is NP-hard.

} X can be used to solve every other problem in NP.



Thm. If X is NP-complete, then X is solveable in poly time iff $P = NP$.

proof. If $P = NP$, then $X \in NP = P$, so X is solveable in poly-time.

Suppose $X \in P$. Then $\forall Y \in NP$, $Y \leq_p X$ so we can solve Y by reducing it in polytime to X, which we can also solve in poly-time, so we can solve Y in poly-time.

$\Rightarrow Y \in P$, so $P = NP$.

Thm If Y is NP-complete, $X \in NP$, and $Y \leq_p X$, then X is NP-complete.

proof. Let $Z \in NP$. Since Y is NP-complete, $Z \leq_p Y \leq_p X$ ▭

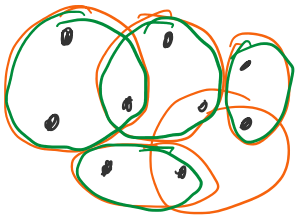
Fact: Vertex Cover is NP-complete.

(easy to verify Vertex Cover is in NP because checking a proposed vertex cover is easy)

$\Rightarrow$ Ind Set is NP-complete.

Problem (Set Cover) Given a set U of elements, and a collection $S_1, \ldots, S_m$ of subsets of U, is there a collection of at most k of these subsets whose union is U?

Set cover



subsets

Lemma: Set Cover $\in$ NP.

proof. The certificate is a list of k sets from $S_1, \ldots, S_m$. We can take the union of these sets and check that the union is equal to U in poly-time ▭

Lemma: Vertex Cover $\leq_p$ Set Cover.

proof- Let $G = (V, E)$ and k be an instance of Vertex Cover.

Create an instance of Set Cover:

- $U = E$
- Create a S_u $\forall u \in V$ where S_u contains the edges adjacent to u.

U can be covered by $\leq k$ sets iff G has a vertex cover of size $\leq k$.

Why? If k sets $S_{u_1}, \ldots, S_{u_k}$ cover U, then every edge is adj to at least one of vertices $u_1, \ldots, u_k$, giving a vertex cover.

Conversely, if $u_1, \ldots, u_k$ is a vertex cover, then sets $S_{u_1}, \ldots, S_{u_k}$ cover U.

$\Rightarrow$ Set Cover is NP-hard

$\Rightarrow$ Set Cover is NP-complete.

Punchline: If you can reduce

Vertex Cover
Ind. Set
or Set Cover

} known NP-complete

to some problem X, then X is NP-hard,
and if additionally, $X \in NP$, then X is NP-complete.

and if additionally, $X \in NP$, then X is NP-complete.

careful about direction of reduction