

Master Theorem (Simplified)

Given a recurrence of form

$$T(n) = aT\left(\frac{n}{b}\right) + \Theta(n^i), \quad \text{constants } a \geq 1, b > 1,$$

simplified, rather than dealing with generic $f(n)$.

Case 1: $i < \log_b a$ then $T(n) = \Theta(n^{\log_b a})$ work increasing as we go down the tree
($a > b^i$)

Case 2: $i = \log_b a$ then $T(n) = \Theta(n^i \log n)$ work same at each level
($a = b^i$)

Case 3: $i > \log_b a$ then $T(n) = \Theta(n^i)$ work decreasing as we go down the tree
($a < b^i$)

Ex $T(n) = 2T\left(\frac{n}{2}\right) + cn \Rightarrow T(n) = \Theta(n \log n)$ (merge sort)

Ex $T(2^k) = 3T\left(\frac{2^k}{2}\right) + c2^k$
 $T(n) = 3T\left(\frac{n}{2}\right) + cn \Rightarrow T(n) = \Theta(n^{\log_2 3}) = \Theta(n^{1.58...})$ (Karatsuba)

Ex $T(n) = 7T\left(\frac{n}{2}\right) + cn^2 \Rightarrow T(n) = \Theta(n^{\log_2 7}) = \Theta(n^{2.807...})$

We're not going to prove the Master Theorem in class, but instead, we're going to prove it for a few special cases of interest.

On an exam, if we ask a question about the Master Thm, we will give it to you, but we might ask you to directly prove that it works in a problem. Or we might ask you to just apply it directly. Both are fair game

Example of divide + conquer for numerics

(Anatoly, 1962)

Karatsuba's alg for integer multiplication

$$\begin{array}{r} 123 \\ \times 321 \\ \hline 123 \\ 246 \\ 369 \\ \hline 39483 \end{array}$$

$$\begin{array}{r} 10101110 \\ \times 01011101 \\ \hline 10101110 \\ \text{---}00000000\text{---} \\ 10101110 \\ 10101110 \\ 10101110 \\ \text{---}00000000\text{---} \\ 10101110 \\ \hline 11111100110110 \end{array}$$

$n = \text{length of numbers}$

numbers of n bits
 $O(n^2)$ -time

Break items into half-sized blocks:

Say x is n bits. Let $m = \frac{n}{2}$.

$$x = x_1 2^m + x_0$$

$$y = y_1 2^m + y_0$$

Then $xy = (x_1 2^m + x_0)(y_1 2^m + y_0) = \underbrace{x_1 y_1}_{\substack{\text{4 mults of size } m \\ \text{bitshift}}} 2^{2m} + \underbrace{(x_1 y_0 + x_0 y_1)}_{\substack{\text{2 mults of size } m \\ \text{bitshift}}} 2^m + x_0 y_0$

But, $x_1 y_0 + x_0 y_1 = (x_1 + x_0)(y_1 + y_0) - x_1 y_1 - x_0 y_0$
one mult

Only 3 mults needed!

Let $p_0 = x_0 y_0$

$p_1 = x_1 y_1$

$p_2 = (x_1 + x_0)(y_1 + y_0) - p_0 - p_1$

Then $xy = p_1 2^{2m} + p_2 2^m + p_0$

linear work at each level of divide + conquer

Analysis: Let $n = 2^k$ for some k .

$$\frac{T(2^k)}{3^k} = \frac{3T(2^{k-1})}{3^k} + \frac{c2^k}{3^k} \quad \left\{ \begin{array}{l} \frac{T(2^k)}{3^k} = \frac{T(2^{k-1})}{3^{k-1}} + c \cdot \left(\frac{2}{3}\right)^k \end{array} \right.$$

By recursion, $\frac{T(2^k)}{3^k} = \frac{T(2^{k-1})}{3^{k-1}} + c \cdot \left(\frac{2}{3}\right)^k$
 $= \frac{T(2^{k-2})}{3^{k-2}} + c \cdot \left(\frac{2}{3}\right)^{k-1} + c \cdot \left(\frac{2}{3}\right)^k$
 $= \dots = \frac{T(1)}{1} + c \sum_{j=1}^k \left(\frac{2}{3}\right)^j \leq \beta$ for some constant β
base case geom series

$\Rightarrow \frac{T(2^k)}{3^k} \leq \beta$

$\Rightarrow T(2^k) \leq \beta 3^k = \beta 2^{\log_2 3^k} = \beta 2^{k \log_2 3}$

$T(n) \leq \beta (2^k)^{\log_2 3} = \beta n^{1.58...} = O(n^{1.58...}) \leq O(n^2)$

Matrix multiplication:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 5+2 \cdot 7 & 6+2 \cdot 8 \\ 5 \cdot 3+4 \cdot 7 & 3 \cdot 6+4 \cdot 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

row i $\begin{bmatrix} a_{i1} & \dots & a_{in} \end{bmatrix}$ $\begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix}$ col j $= \begin{bmatrix} \sum_{k=1}^n a_{ik} b_{kj} \end{bmatrix}$
 $n \times n$ $n \times n$ $n \times n$
each entry takes $\Theta(n)$ to compute
 n^2 entries $\Rightarrow \Theta(n^3)$

Notice: Matrix addition is much cheaper than multiplication

Let's try divide (conquering for later)

Assume matrix is of size $2^m \times 2^m$ for some m (so we can break the matrix into quarters)

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

$A \in \mathbb{R}^{n \times n}$ $B \in \mathbb{R}^{n \times n}$ $C \in \mathbb{R}^{n \times n}$

$C_{11} = A_{11} B_{11} + A_{12} B_{21}$ $C_{12} = A_{11} B_{12} + A_{12} B_{22}$ 8 half-size matrix multiplications
 $C_{21} = A_{21} B_{11} + A_{22} B_{21}$ $C_{22} = A_{21} B_{12} + A_{22} B_{22}$ ($2^3 = 8$)

Intuition: $(X+Y)(Z+W) = \underbrace{XZ}_{1 \text{ mult}} + \underbrace{YZ}_{1 \text{ mult}} + \underbrace{XW}_{1 \text{ mult}} + \underbrace{YW}_{1 \text{ mult}}$
Can we rewrite the formulae for $C_{11}, C_{12}, C_{21}, C_{22}$ to use fewer mults

Strassen:

$P_1 = (A_{11} + A_{22})(B_{11} + B_{22})$
 $P_2 = (A_{21} + A_{22})B_{11}$ $C_{11} = P_1 + P_4 - P_5 + P_7$
 $P_3 = A_{11}(B_{12} - B_{22})$ $C_{12} = P_3 + P_5$
 $P_4 = A_{22}(B_{21} - B_{11})$ $C_{21} = P_2 + P_4$
 $P_5 = (A_{11} + A_{12})B_{22}$ $C_{22} = P_1 - P_2 + P_3 + P_6$
 $P_6 = (A_{21} - A_{11})(B_{11} + B_{22})$
 $P_7 = (A_{12} - A_{22})(B_{21} + B_{22})$

only 7 mults instead of 8

Apply idea recursively to $P_1, P_2, \dots, P_7$ which are square matrices half the size

Recurrence: $T(n) = 7T\left(\frac{n}{2}\right) + cn^2$
 $n = 2^m$: $\frac{T(2^m)}{7^m} = \frac{7T(2^{m-1})}{7^m} + \frac{c4^m}{7^m}$
 $\Rightarrow \frac{T(2^m)}{7^m} = \frac{T(2^{m-1})}{7^{m-1}} + c\left(\frac{4}{7}\right)^m$
Recursive expansion:
 $= \sum_{k=1}^m c\left(\frac{4}{7}\right)^k + \frac{T(1)}{1}$
 $\leq c \sum_{k=1}^{\infty} \left(\frac{4}{7}\right)^k + \frac{T(1)}{1} = c \cdot \frac{\frac{4}{7}}{1 - \frac{4}{7}} + T(1) = \frac{4}{3}c + T(1) = \alpha$ for some constant α .
 $\Rightarrow \frac{T(2^m)}{7^m} \leq \alpha \Rightarrow T(2^m) \leq \alpha 7^m = \alpha 2^{m \log_2 7}$
 $\Rightarrow T(n) \leq \alpha n^{\log_2 7} = \alpha n^{2.807...} = O(n^{2.807...})$

Strassen first to show improvement over $O(n^3)$

Can we do better?

Volker Strassen, 1969, $O(n^{2.807})$
 $\vdots$
Coppersmith + Winograd, 1990, $O(n^{2.3754})$
Andrew Stothers, 2010, $O(n^{2.3736})$
Virginia Williams, 2013, $O(n^{2.3729})$ (CMU, PhD, 2008)
 $\vdots$
Williams, Xu, Zhao, 2023, $O(n^{2.371552})$