Lec22-max-flow-min-cut KT 7.1

Sunday, November 3, 2024 11:00 PM

Suppose are Andrew Carnegie, and you want to ship steel from Pittsburg to Washington DC.

Each railroad track has a capacity.

How can you send as much steel as possible



A flow network is a connected, directed graph $G=(V,E)$
- each edge has a capacity $c_e \in \mathbb{N}$. (you integers)
- source vertex $s \in V$ (no in-edges)
- sink vertex $t \in V$ (no out-edges)

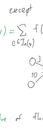
An s-t flow is a function $f: E \to \mathbb{R}^{\geq 0}$ representing amt of material carried on each edge, where

$0 \leq f_e \leq c_e$

$\forall v \in V$ except s & t, we have

$f^{in}(v) = \sum_{e \in In(v)} f(e) = \sum_{e \in Out(v)} f(e) = f^{out}(v)$ ← in- and out-edges of v respectively

Ex.



The value of flow f is $v(f) = \sum_{e \in Out(s)} f(e) = f^{out}(s)$ total amt produced

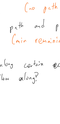
Problem (Maximum flow): Given a flow network G, find a flow f to maximize $v(f)$.

Algorithm idea: (greedy start)

1. Let $f(e)=0 \quad \forall e \in E$.
2. Repeat until stuck (no path with remaining capacity)
   Choose an s-t path and push maximum flow possible along it (min remaining capacity edge along path)
3. Undo some flow along certain edges to create more paths to push flow along?



$v(f) = 20$



No more path left. What if we undo 10 units of w→v flow?

What:



$v(f) > 10$

Residual graph

Given flow f on graph G, define the residual graph G_f on the same nodes but modified flow edges

Forward edge: $\forall e=(u,v) \in G$ where $f(e) < c_e$, include edge $e'=(u,v) \in G_f$ with capacity $c_e - f(e)$ } capacity remaining

Backward edge: $\forall e=(u,v) \in G$ where $f(e) > 0$, include edge $e'=(v,u) \in G$ with capacity $f(e)$ } can erase allocated flow
(If this is already existing edge, add to capacity)

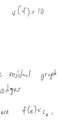


flow f



residual graph G_f

path P with capacity 10 in residual graph



augmented flow $f+P$.

If P is an s-t path in G_f with smallest capacity edge bottleneck$(f,P)>0$, we can create new flow $f+P$ by sending bottleneck(f,P) along P. (increases value of f)

augment(f,P):
  b = bottleneck(P,f)
  $\forall e=(u,v) \in P$:
    If e is a forward edge: (with capacity $c_e - f(e)$)
      Increase $f(e)$ by b ($f(e) \colon f(e) + b \leq f(e) + c_e - f(e) = c_e$)
    If e is a backward edge: (has capacity $f(e)$)
      Let $e' = (v,u)$.
      Decrease $f(e')$ by b ($f(e') \geq 0$)
  return f.

Capacity constraint still met:





Ford-Fulkerson alg

Max-Flow (G):
  Set $f(e)=0 \quad \forall e \in G$
  While $P \leftarrow$ Find Path$(s,t,$ Residual$(G,f)) \neq$ None:
    $f \leftarrow$ augment(f,P)
  Return f

Running time:
- All values are integers
- Always augment value(f) by at least 1 unit
- Never send more than $C = \sum_{e \text{ out of } s} c_e$.

$\Rightarrow$ Ford-Fulkerson terminates in at most C iterations of augmenting.

- G has an edge $\Rightarrow G_f$ has $\leq 2m$ edges
- Can find s-t path in G_f in $O(n+m)$ time
  WLOG, $n \leq O(m)$ because any node without an edge can be removed.

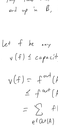
$\Rightarrow$ Ford-Fulkerson is $O(mC)$.

Minimum cut

Recall that a cut of a graph $G=(V,E)$ is a partition of the nodes in two parts A, B, (since $B = V-A$)
The weight of a cut $(A,B) = \sum_{\substack{e \text{ out of } A \\ \text{into } B}} w(e)$.

In a flow network, an s-t cut (A,B) is a cut s.t. $s \in A$, $t \in B$.
The capacity of an s-t cut $(A,B) = \sum_{\substack{e \text{ out of } A \\ \text{into } B}} c_e$

Thm: Let f be an s-t flow and (A,B) an s-t cut.
Then $v(f) = f^{out}(A) - f^{in}(A)$



pf: Any flow that leaves A must either return to A or end up in B, but not both. It needs to if B must end with t.

Thm: Let f be any s-t flow and (A,B) be any s-t cut.
Then $v(f) \leq$ capacity(A,B)

pf: $v(f) = f^{out}(A) - f^{in}(A)$
$\leq f^{out}(A)$
$= \sum_{e \text{ out of } A} f(e) \leq \sum_{e \text{ out of } A} c_e =$ capacity(A,B)

$\Rightarrow$ all cuts are at least as big as any flow

$\Rightarrow$ minimum cut constrains maximum flow capacity

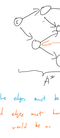
Story: Soviet scientists like A.N. Tolstoi [1930] wanted to maximize flow of goods they created on Soviet railways. That is a max-flow problem.

The US Air Force was also studying Soviet railways, but because they wanted to break it [1950, Ellis, Harris]. This is a min-cut problem.

Turns out both problems are "dual" to each other.

Max Cut = Min Flow

Consider s-t returned by Ford-Fulkerson.
Then G_{f^*} defines a cut in G by the nodes reachable from s A^* and the nodes not reachable from s B^*.
(If s-t path existed, we could increase flow, so f^* is max flow.)



Blue edges must be saturated. (or else in G_{f^*} that edge is present)
Red edges must have 0 flow. (or else in G_{f^*} there would be an opposite direction edge connecting A^* & B^*)

$\Rightarrow v(f^*) =$ capacity(A^*, B^*)

$\Rightarrow$ No flow can have value bigger than capacity(A^*, B^*)

$\Rightarrow f^*$ is a max-flow & (A^*, B^*) is a min-cut.

Thm (Max-flow Min-cut): The value of the maximum flow in any flow graph is equal to the capacity of the minimum cut.

Can find a minimum cut by constructing max flow f^* using BFS to find nodes reachable in G_{f^*}.

Ford-Fulkerson only is guaranteed to terminate with integer weights.

Variation Edmonds-Karp works for any weights in $O(VE^2)$ time.