Lec25-simplex-method

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Linear programming

maximize $\vec{c} \cdot \vec{x}$ ($\vec{c} \in \mathbb{R}^n$, $\vec{x} \in \mathbb{R}^n$)

subject to $A\vec{x} \le \vec{b}$ ($A \in \mathbb{R}^{m \times n}$, $\vec{b} \in \mathbb{R}^m$)

and $\vec{x} \ge 0$

Minimization, "≥", & "=" can be reduced to this

standard form when we add constraint that all variables are positive

Notice: given a problem with potentially negative variables, we can replace with two positive variables.

constraints
$$\begin{cases} 2x_1 + x_2 \le 10 \\ -x_1 \le -1 \end{cases} \Rightarrow \begin{aligned} 2x_1 + \underbrace{x_2^+ - x_2^-}_{} &\le 10 \\ -x_1 &\le -1 \end{aligned}, \text{ where } x_2^+, x_2^- \ge 0.$$

x_1 already positive

obj: $x_1 + x_2$ obj: $x_1 + x_2^+ - x_2^-$

$\underbrace{x_2^+ - x_2^-}_{\text{any pos or neg number}} = x_2 \in \mathbb{R}$

Thus, the above standard form works for all LP.

Idea behind simplex: Increase objective by repeatedly turning up a variable until it can no longer be increased.

Example simplex run

All $s_i \ge 0$, slack variables

Constraints
$$\begin{cases} x_1 + x_2 \le 3 \\ -x_1 + 3x_2 \le 1 \\ x_2 \le 3 \end{cases} \longrightarrow \begin{aligned} x_1 + x_2 + s_1 &= 3 \\ -x_1 + 3x_2 + s_2 &= 1 \\ x_2 + s_3 &= 3 \end{aligned}$$

Maximize $x_1 + x_2 = z$ Maximize $x_1 + x_2 = z$

Also, all $x_i \ge 0$.

easy to find a solution (initial feasible) to constraints, but equivalent to optimize.

Set $s_1 = 3$, $s_2 = 1$, $s_3 = 3$; $x_1 = 0$, $x_2 = 0$

(harder when RHS is negative, will see later)

Tableau (RHS vars $= 0$) (LHS may be nonzero)

$$\begin{aligned} s_1 &= 3 - x_1 - x_2 \\ s_2 &= 1 + x_1 - 3x_2 \\ s_3 &= 3 - x_2 \\ \hline z &= 0 + x_1 + x_2 \end{aligned}$$

} To increase z, we want to increase a variable with pos. coefficient

Pick x_2 as "entering variable"

Variables on LHS must also change in value, but can't be negative

$$\begin{cases} s_1 = 3 - x_2 \ge 0 \\ s_2 = 1 - 3x_2 \ge 0 \\ s_3 = 3 - x_2 \ge 0 \end{cases} \Bigg\} \begin{aligned} &x_2 \le 3 \\ &\boxed{x_2 \le \tfrac{1}{3}} \\ &x_2 \le 3 \end{aligned}$$

← strictest inequality

choose s_2 as "leaving variable"

set $x_2 = \frac{1}{3}$

$s_2 = 1 + x_1 - 3x_2$

$x_2 = \frac{1}{3} + \frac{1}{3}x_1 - \frac{1}{3}s_2$

rewrite tableau with x_2 on LHS, s_2 on RHS

$$\begin{aligned} s_1 &= 3 - x_1 - \left(\tfrac{1}{3} + \tfrac{1}{3}x_1 - \tfrac{1}{3}s_2\right) = \tfrac{8}{3} - \tfrac{4}{3}x_1 + \tfrac{1}{3}s_2 \\ x_2 &= \tfrac{1}{3} + \tfrac{1}{3}x_1 - \tfrac{1}{3}s_2 \\ s_3 &= 3 - \left(\tfrac{1}{3} + \tfrac{1}{3}x_1 - \tfrac{1}{3}s_2\right) = \tfrac{8}{3} - \tfrac{1}{3}x_1 + \tfrac{1}{3}s_2 \\ \hline z &= x_1 + \left(\tfrac{1}{3} + \tfrac{1}{3}x_1 - \tfrac{1}{3}s_2\right) = \tfrac{1}{3} + \underbrace{\tfrac{4}{3}x_1 - \tfrac{1}{3}s_2}_{\text{to increase } z \text{, increase a variable with positive coefficients}} \end{aligned}$$

} objective now $\frac{1}{3}$

Choose x_1 as leaving var.

Find strictest constraint:

$$\left.\begin{aligned} \tfrac{8}{3} - \tfrac{4}{3}x_1 &\ge 0 \\ \tfrac{1}{3} + \tfrac{1}{3}x_1 &\ge 0 \\ \tfrac{8}{3} - \tfrac{1}{3}x_1 &\ge 0 \end{aligned}\right\} \begin{aligned} &\boxed{x_1 \le 2} \\ &x_1 \ge -1 \\ &x_1 \le 8 \end{aligned}$$

← strictest constraint

← not a constraint at all because x_1 had pos. coeff.

$$s_1 = \tfrac{8}{3} - \tfrac{4}{3}x_1 + \tfrac{1}{3}s_2 \Rightarrow x_1 = 2 - \tfrac{3}{4}s_1 + \tfrac{1}{4}s_2$$

Substitute in x_1 to all equations

$$\begin{aligned} x_1 &= 2 - \tfrac{3}{4}s_1 + \tfrac{1}{4}s_2 \\ x_2 &= 1 - \tfrac{1}{4}s_1 - \tfrac{1}{4}s_2 \\ s_3 &= 2 + \tfrac{1}{4}s_1 + \tfrac{1}{4}s_2 \\ \hline z &= 3 \underbrace{- s_1}_{\text{done since all vars have neg coeff.}} \end{aligned}$$

} Note: increasing x_1 also increased x_2

Optimal value is $z = 3$ with $x_2 = 2$, $x_1 = 1$.

Can check that this satisfies original constraints.

Summary of simplex so far

1. Write LP with slack variables, & get initial sol.
2. Choose variable v in obj with pos coeff. to increase
3. Choose the strictest constraint (will be one where v has neg. coeff.)
4. Rewrite strictest eq with v on LHS, & substitute for v everywhere else
5. If all coeff in obj are neg, done. Else, go back to step 2.

pivot: variables on LHS are "basic" variables.
each pivot swaps a different variable in as a basic var while improving (or at least not decreasing) obj.
(may change value of other basic vars)

If no constraint provides upper bound, then problem is unbounded.

If strictest upper bound is 0, then ∃ other feasible sol of equiv. cost.
⇒ may have to pivot w/o increasing obj to make progress. (or decreasing)

Pivot rules:

- largest coeff in obj ← max increase per unit of entering var when ignoring other vars
- largest increase after pivot in obj
- random
- steepest edge $\dfrac{\vec{c} \cdot (\vec{x}_{new} - \vec{x}_{old})}{\|\vec{x}_{new} - \vec{x}_{old}\|}$ } max increase per unit of change in overall $\vec{x}$ vector
- Bland's rule - choose entering var w/ lowest index (and leaving var with lowest index)

↳ prevents infinite cycling. Not practical, but important theoretically.
almost never happens even w/ other rules, but not guaranteed like it is w/ Bland.

Optimality

The final objective is equivalent in value to original.
And is always (e.g. $z = 24 \underbrace{- 5x_1 - 3x_3}_{\text{neg coeff}}$) with only neg coeff., so we can't do better by any more pivots.

Initialization

We assumed above that $b_i \ge 0$, so we could set non-slack vars to 0 and slack vars to b_i, for an initial feasible sol.

What if $b_i < 0$?

Ex

$$\begin{aligned} s_1 &= 3 - x_1 - x_2 \\ s_2 &= -1 + x_1 - 3x_2 \\ s_3 &= 3 - x_2 \\ \hline z &= 0 + x_1 + x_2 \end{aligned}$$

← Can't set $s_2 = -1$.

Auxiliary problem

ignore original opt

Given LP max $\vec{c} \cdot \vec{x}$ s.t. $A\vec{x} \le \vec{b}$, $\vec{x} \ge 0$,

Auxiliary LP max $-y$ s.t. $A\vec{x} - y \le \vec{b}$, $\vec{x} \ge 0$, $y \ge 0$

new var y

Introduce slack variables $s_i \ge 0$ as usual, but for the initial feasible sol, set $x_i = 0$, $y = \min_i b_i$, and $s_i = b_i - y$, all non-negative.

In auxiliary LP, we are trying to get rid of y.
- if we can't, then no feasible sols
- if we do, then we have feasible sol to original LP, and can continue the tableau from there, dropping all mention of y (which is 0)