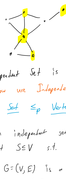
Lec28-more-reductions

Thursday, November 21, 2024 11:56 AM

Vertex Cover

Def A vertex cover of graph $G(V,E)$ is a set $S \subseteq V$ s.t. $\forall e \in E$, at least one endpoint of e is in S

Ex



Fact: Independent Set is NP-hard.

We can now use Independent Set to prove Vertex Cover is NP-hard

Independent Set $\leq_p$ Vertex Cover

Recall: An independent set in a graph $G=(V,E)$ is a set $S \subseteq V$ s.t. $\forall u,v \in S$, $(u,v) \notin E$.

Thm If $G=(V,E)$ is a graph, then S being an ind set $\Leftrightarrow$ $V-S$ is a vertex cover.

proof ($\Rightarrow$) Suppose S is an independent set, and let $e=(u,v) \in E$

Only one of u,v can be in S

$\Rightarrow$ At least one of u,v is in $V-S$.

$\Rightarrow V-S$ is a vertex cover.

($\Leftarrow$) Suppose $V-S$ is a vertex cover, and let $u,v \in S$

$\Rightarrow (u,v) \notin E$, as that edge isn't covered by $V-S$

$\Rightarrow S$ is an ind set.

So vertex covers and independent sets are complements.

Given an instance of Independent Set $\langle G, k \rangle$, we ask our Vertex Cover black box if $\exists$ a vertex cover C of size $\leq |V|-k$.

If so, we define $S = V-C$ as our ind set for Independent Set, as S is an ind set of size k.

If not, then no vertex cov of size $\leq |V|-k$, so no independent set of size $\geq k$.

$\Rightarrow$ Ind Set $\leq_p$ Vertex Cover

Lemma: Vertex Cover $\in$ NP

proof Given a proposed vertex cover, can check in poly time if it is actually a cover by checking every edge. ☑

$\Rightarrow$ Vertex Cover is NP-complete.

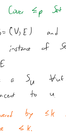
Claim: Vertex Cover $\leq_p$ Ind Set

Two options: (1) Vertex Cover $\leq_p$ 3SAT $\leq_p$ Ind Set

(2) Construct a direct reduction (in recitation)

$\Rightarrow$ Vertex Cov & Ind Set are equivalently difficult.

Problem (Set Cov) Given a set U of elements and a collection $S_1, \dots, S_m$ of subsets of U, is there a collection of at most k of these subsets whose union is U?



Lemma: Set Cover $\in$ NP

proof The certificate is a list of k sets from $S_1, \dots, S_m$

We can take the union of these sets and check that the union is equal to U in poly-time. ☑

Lemma: Vertex Cover $\leq_p$ Set Cover

proof Let $G=(V,E)$ and k be an instance of Vertex Cover

Create an instance of Set Cover:

- $U = E$
- Create a S_v for $v \in V$ where S_v contains the edges adjacent to v

U can be covered by $\leq k$ sets iff G has a vertex cover of size $\leq k$.

Why? If k sets $S_{v_1}, \dots, S_{v_k}$ cover U, then every edge is adj to at least one of vertices $v_1, \dots v_k$, giving a vertex cover.

Conversely, if $v_1, \dots v_k$ is a vertex cover, then sets $S_{v_1}, \dots, S_{v_k}$ cover U.

☑

$\Rightarrow$ Set Cover is NP-hard

$\Rightarrow$ Set Cover is NP-complete

Hamiltonian Path: Does G contain a path that visits every node exactly once? (doesn't have to return to start)

We could adapt the 3SAT $\rightarrow$ HamCycle reduction

Or, we can instead show HamCycle $\rightarrow$ 3SAT.

Thm HamPath is NP-complete.

proof HamPath $\in$ NP because easy to check path

Will show HamCycle $\leq_p$ HamPath

Given HamCycle instance G, choose arbitrary node v and split it into two nodes v^{in}, v^{out}, in graph G'.



Any Ham Path must start at v^{out} & end at v^{in}.

G' has HamPath $\Leftrightarrow$ G has HamCycle

because if G' has HamPath, some path after v^{out} & v^{in} are "glued together" in HamCycle, and if G has HamCycle, that provides a HamPath in G' by using the split above.

Traveling Salesman Problem

Given n cities, distances $d(i,j)$ between cities, does there exist a path of length $\leq k$ that visits each city and returns home?

Note: $d(i,j) \neq d(j,i)$ in general

And: $d(i,j) \leq d(i,k) + d(k,j)$ does NOT hold. (no $\triangle$ inequality)

Thm TSP is NP-complete.

pf: TSP $\in$ NP because if we are given a path, we can easily check that it visits every city & compute the length.

Next, we will show that HamCycle $\leq_p$ TSP.

Need to relate HamCycle to TSP instances.

HamCycle:

Input: Graph $G=(V,E)$

What: cycle that visits every city exactly once & returns to start

TSP Instance D:

city c_i for every vertex v_i.

Let $d(c_i, c_j) = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 2 & \text{otherwise} \end{cases}$

Lemma: G has HamiltonIan cycle $\Leftrightarrow$ D has tour of length $\leq n$.

proof. If G has Ham. Cycle, then this ordering of cities gives a tour of length $\leq n$ in D (only distances of length 1 used)

Suppose D has a tour of length $\leq n$.

Then the tour can't ever use a dist-2 connections, because with n steps, that would be too long, so it visits cities only via dist-1 connections, which are exactly edges in G, giving a Ham. Cycle.

Thus, HamCycle $\leq_p$ TSP.

$\Rightarrow$ TSP is NP-hard.

Since TSP $\in$ NP & is NP-hard,

TSP is NP-complete.

Even if distances are Euclidean, TSP is NP-complete.