15-351 / 15-650 / 02-613: Exercises 3c

For any following NP-completeness proof problems, you can only assume that Vertex Cover, Set Cover, Independent Set, 3-SAT, Hamiltonian Cycle, and Hamiltonian Path, Traveling Salesman Problem, which we discussed in class, are NP-complete.

1. **Fewest suitcases.** Suppose we are given a set of n objects, where the weight of object i is s_i, with $0 < s_i < 1$. We want to put these n objects into a series of suitcases, each of which can hold a total of 1 unit of weight. We have as many suitcases as we want, but we want to use the *fewest* suitcases. This is an NP-hard problem.

   The "first-fit" algorithm: process the objects and suitcases in arbitrary order and put it in the first suitcase that can still hold it. If no suitcase can, grab a new suitcase and place it in that.

   (a) Let $S = \sum_{i=1}^{n} s_i$. Show that the optimal number of suitcases is $\geq \lceil S \rceil$.

   (b) Prove that first-fit never uses more than $\lceil 2S \rceil$ suitcases.

   (c) Explain why this shows that first-fit is a 2-approximation algorithm for this problem.

2. **How Universal?** In the lecture, we talked about universal hash functions. There is an extension of this idea to L-universality. Specifically, a function is called L-universal if for every set of L distinct keys $x_1, x_2, \ldots, x_L$ and every set of L values $v_1, v_2, \ldots, v_L \in \{0, \ldots, M-1\}$, we have

$$\Pr_{h \leftarrow H}[h(x_1) = v_1 \text{ AND } h(x_2) = v_2 \text{ AND } \ldots \text{ AND } h(x_L) = v_L] = 1/M^L.$$

   It is easy to see that if H is 2-universal then it is universal. Note that the matrix-based hash family that we saw in the lecture was not 2-universal since the hash functions all mapped 0 to 0.

   Show that if we choose $A \in \{0,1\}^{m \times u}$ (where u is the key length and $M = 2^m$) and $b \in \{0,1\}^m$ independently and uniformly at random, then the hash family $h(x) = Ax + b$ is 2-universal.

3. **Hard to approximate.** Prove that the problem of finding a k-approximation of the TSP is NP-hard for any constant $k > 1$. In other words, show that if you are given a black box that returns a k-approximation of TSP for a graph (i.e. a tour of total length $\leq k \cdot OPT$), you can use that black box to solve another known NP-hard problem. Hint: reduce from Hamiltonian Cycle.