Problem Set 2

[Your name] and [student ID]
MAT1801-2020

Problem 1 [BHK 3.22] (10 points).

1. For any matrix A, show that $\sigma_k \leq \frac{\|A\|_F}{\sqrt{k}}$.
2. Prove that there exists a matrix B of rank at most k such that $\|A - B\| \leq \frac{\|A\|_F}{\sqrt{k}}$.
3. Can the 2-norm of the left hand side in (2.) be replaced by the Frobenius norm?

Problem 2 [BHK 3.23] (10 points). Suppose an $n \times d$ matrix A is given and you are allowed to preprocess A. Then you are given a number of d-dimensional vectors $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_m}$ and for each of these vectors you must find the vector $A\mathbf{x_j}$ approximately, in the sense that you must find a vector $\mathbf{y_j}$ satisfying $|\mathbf{y_j} - A\mathbf{x_j}| \leq \epsilon \|A\|_F |\mathbf{x_j}|$. Here, $\epsilon > 0$ is a given error bound. Describe an algorithm that accomplishes this in time $O\left(\frac{d+n}{\epsilon^2}\right)$ per $\mathbf{x_j}$ not counting the preprocessing time. Hint, use Problem 1.

Problem 3 [BHK 3.27] (10 points). Read in a photo and convert to a matrix. Perform a singular value decomposition of the matrix. Reconstruct the photo using only 5%, 10%, 25%, 50% of the singular values.

1. Print the reconstructed photo. How good is the quality of the reconstructed photo?
2. What percent of the Frobenius norm is captured in each case?

Hint: You may choose to use a greyscale image, as then you won't have to deal with all three color channels (it's not that much harder to deal with 3 color channels, but it's up to you). If you are using Python, you may wish to use imageio.imread() https://imageio.readthedocs.io/en/stable/examples.html. You may use the built in SVD libraries in Python.