14. Random graphs and phase transitions

Wednesday, October 13, 2021 3:05 PM

Last time: Finished up section on probabilistic sketching

Today: Erdős-Renyi random graphs + phase transitions

Erdős-Renyi random graphs

Let $G(n,p)$ be a graph-valued random variable, with vertices V and edges E.

- $n = |V|$
- $p = \text{Prob}\left((v_i, v_j) \in E \right)$ for any $i \neq j$ (prob of each edge is ind.)

Thm 8.1 Let $v_1 \in V$ of the random graph $G(n,p)$. Aside: $\mathbb{E} \deg(v_i) = p(n-1)$

Let $\alpha \in (0, \sqrt{np})$. Then

$$\text{Prob}\left(|(n-1)p - \deg(v_1)| \geq \alpha \sqrt{(n-1)p} \right) \leq 3e^{-\frac{\alpha^2}{8}}$$

proof. $\deg(v_1) = \sum_{i=2}^{n} \mathbb{1}_{1,i}$, where $\mathbb{1}_{1,i} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{else} \end{cases}$

Then the proof follows from Chernoff bounds (Thm 12.6)

$$\text{Prob}\left(|\deg(v_1) - (n-1)p| \geq c(n-1)p \right) \leq 3e^{-(n-1)pc^2/8}$$

let $c = \dfrac{\alpha}{\sqrt{(n-1)p}}$ $\square$

Corollary 8.2 Suppose $\varepsilon > 0$. If $p \geq \frac{9 \ln n}{(n-1)\varepsilon^2}$, then with $1 - o(1)$ probability

$$\deg(v_i) \in \left[(1-\varepsilon)(n-1)p, (1+\varepsilon)(n-1)p \right] \quad \forall i.$$

proof. Let $\alpha = \varepsilon\sqrt{(n-1)p}$ in Thm 8.1.

Then for a given i, failure prob is $\leq 3e^{-\varepsilon^2(n-1)p/8}$

By union bound, failure prob. for ~~any~~ any i $\leq 3ne^{-\varepsilon^2(n-1)p/8}$

$p \geq \frac{9 \ln n}{(n-1)\varepsilon^2}$ $\quad \leq 3ne^{-\frac{9}{8}\ln n} = 3n \cdot n^{-9/8} = 3n^{-\frac{1}{8}} = o(1)$ $\square$

i.e. If $\underbrace{p = \Omega\left(\frac{\log n}{n}\right)}$, then with vanishing prob, all vertices have tightly concentrated degree

Note $p = \Omega\left(\frac{1}{n}\right)$ fails. look at prob of deg 0 when $p = \frac{1}{n}$.

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Claim: $G\left(n, \frac{d}{n}\right)$ has in expectation $\approx \frac{d^3}{6}$ triangles $\qquad p = \frac{d}{n}$

Moral justification: As n increases, # triples grows with n^3

But each pair has $\frac{d}{n}$ prob. of having an edge, so 3 pairs $= \frac{d^3}{n^3}$ chance

$\frac{d^3}{n^3}$ chance balances out n^3 # triples

proof. Let Δ_{ijk} be the indicator variable for existence of triangle $\underbrace{i, j, k}_{\in V}$.

Then $\mathbb{E}(\#\text{ triangles}) = \mathbb{E}\left(\sum_{ijk} \Delta_{ijk}\right) = \sum_{ijk} \mathbb{E}(\Delta_{ijk}) = \binom{n}{3}\left(\frac{d}{3}\right)^3 = \frac{n(n-1)(n-2)}{6} \cdot \frac{d^3}{n^3}$

linearity of expectation doesn't depend on independence

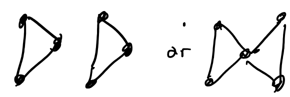
$\approx \frac{d^3}{6}$

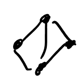
Let's try to bound # of triangles

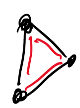
Let $x = \#$ triangles, $x = \sum_{ijk} \Delta_{ijk}$.

$$\mathbb{E}(x^2) = \mathbb{E}\left(\sum_{ijk} \Delta_{ijk}\right)^2 = \mathbb{E}\left(\sum_{\substack{i,j,k \\ i',j',k'}} \Delta_{i,j,k}\, \Delta_{i',j',k'}\right)$$

Split sum into 3 parts:

$S_1 = \{ i, j, k, i', j', k' \mid \Delta_{ijk} \text{ and } \Delta_{i'j'k'} \text{ share no edges} \}$ 

$S_2 = \{ i, j, k, i', j', k' \mid \Delta_{ijk} \text{ and } \Delta_{i'j'k'} \text{ share exactly 1 edge} \}$ 

$S_3 = \{ i, j, k, i', j', k' \mid \Delta_{ijk} = \Delta_{i'j'k'} \}$ 

$$\mathbb{E}\left(\sum_{S_1} \Delta_{ijk} \Delta_{i'j'k'}\right) = \sum_{S_1} \underbrace{\mathbb{E}(\Delta_{ijk})\, \mathbb{E}(\Delta_{i'j'k'})}_{\text{ind. because no edges shared}} \leq \left(\sum_{\text{all } ijk} \mathbb{E}(\Delta_{ijk})\right)\left(\sum_{\text{all } i'j'k'} \mathbb{E}(\Delta_{i'j'k'})\right) = (\mathbb{E}x)^2$$

$$\mathbb{E}\left(\sum_{S_2} \Delta_{ijk} \Delta_{i'j'k'}\right) = \underbrace{\binom{n}{4}}_{\text{ways to choose 4 vertices}} \underbrace{\binom{4}{2}}_{\text{ways to choose 2 vertices lacking an edge}} \underbrace{p^5}_{\text{chance remaining 5 edges are present}} \approx \frac{n^4}{24} \cdot 6 \cdot p^5 = \frac{1}{4} n^4 p^5 = \frac{1}{4} n^4 \cdot \frac{d^5}{n^5}$$

$$= \frac{1}{4} \cdot \frac{d^5}{n} = o(1)$$

$$\mathbb{E}\left(\sum_{S_2} \Delta_{ijk}\Delta_{i'j'k'}\right) = \mathbb{E}\left(\sum_{S_3} \Delta_{ijk}\right) = \mathbb{E}x.$$

$$\Rightarrow \mathbb{E}(x^2) \le (\mathbb{E}x)^2 + \mathbb{E}x + o(1)$$

$$\Rightarrow \mathrm{Var}(x) = \mathbb{E}(x^2) - (\mathbb{E}x)^2 \le \mathbb{E}x + o(1)$$

Then $\mathrm{Prob}(x=0) \le \mathrm{Prob}(|x - \mathbb{E}x| \ge \mathbb{E}x)$

By Chebyshev, $\mathrm{Prob}(x=0) \le \dfrac{\mathrm{Var}(x)}{(\mathbb{E}x)^2} \le \dfrac{\mathbb{E}x + o(1)}{(\mathbb{E}x)^2} \le \dfrac{6}{d^3} + o(1)$

Thus, if $d > \sqrt[3]{6} \approx 1.8$, $\mathrm{Prob}(x=0) < 1$, so $G(n, \frac{d}{n})$ has a triangle with non zero prob.

For $d < \sqrt[3]{6}$, $\mathbb{E}x = \frac{d^3}{6} < 1$, so not many triangles to go around!

Intuitively, need many vertices with $\deg \ge 2$ to have triangles.

Phase transition

Def. If $\exists\, p(n)$ s.t. when $\lim_{n\to\infty} \frac{p_1(n)}{p(n)} = 0$, $G(n, p_1(n))$ lacks a property (almost surely) a.s.

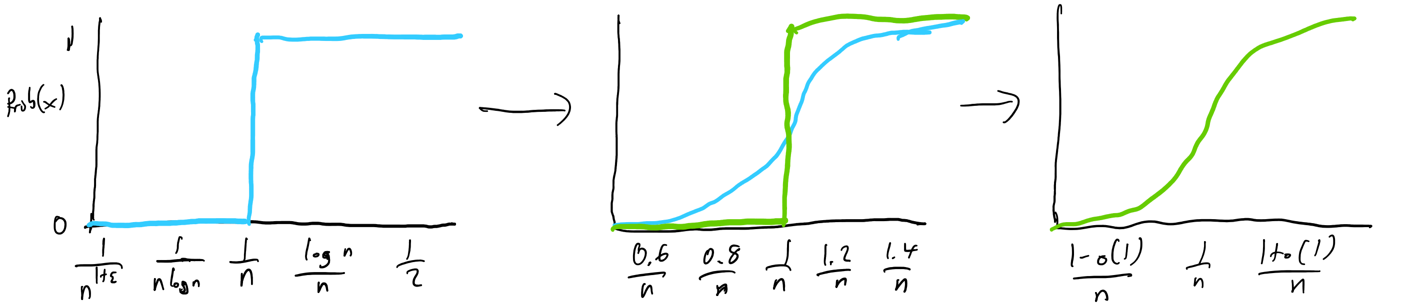
when $\lim_{n\to\infty} \frac{p_2(n)}{p(n)} = \infty$, $G(n, p_2(n))$ has a prop almost surely,

then a phase transition occurs at the threshold $p(n)$.

Def. If for $cp(n)$, $c<1$, $G(n, cp(n))$ lacks a prop. a.s.

$c>1$, $G(n, cp(n))$ has a prop. a.s.

then $p(n)$ is a sharp threshold.

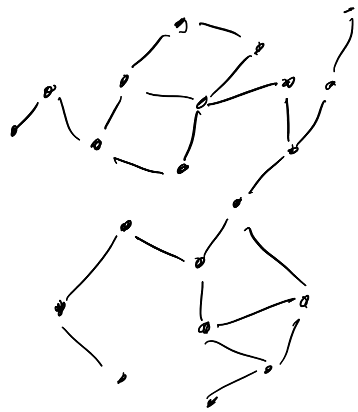


asymp phase trans. at $\frac{1}{n}$

may be smoother when zoomed in unless sharp

If prev. sharp, can zoom in even more.

Phase transitions of Erdős-Renyi graphs



Probability	Behavior
$p = o(\frac{1}{n})$	Forest of trees, component size $O(\log n)$
$p = \frac{d}{n},\ d<1$	Some cycles, component size $O(\log n)$
$p = \frac{d}{n},\ d=1$	Components of size $O(n^{2/3})$
$p = \frac{d}{n},\ d>1$	Giant component + $O(\log n)$ components
$p = \frac{1}{2}\cdot\frac{\ln n}{n}$	Giant component + isolated vertices
$p = \frac{\ln n}{n}$	No isolated vertices, Appearance of Hamiltonian circuit Diameter $O(\log n)$
$p = \sqrt{\frac{2\ln n}{n}}$	Diameter 2
$p = \frac{1}{2}$	Clique of size $(2-\varepsilon)\ln n$.

First-moment method

Let $x(n)$ denote the number of occurences of an item in a random graph.

If $\mathbb{E}x(n) \to 0$ as $n\to\infty$, then a random graph almost surely has no occurences of the item.

proof. Markov's inequality, x is non-negative.

$\text{Prob}(x \geq a) \leq \frac{1}{a}\mathbb{E}x$, so $\text{Prob}(x(n)\geq 1) \leq \mathbb{E}x(n) \to 0$ as $n\to\infty$.

Second-moment method

Let $x(n)$ be a r.v. with $\mathbb{E}x > 0$.

If $\text{Var}(x) = o((\mathbb{E}x)^2)$, then $x > 0$ a.s.

proof. $\text{Prob}(x \leq 0) \leq \text{Prob}(|x - \mathbb{E}x| \geq \mathbb{E}x)$

By Chebyshev, $\leq \dfrac{\text{Var}(x)}{(\mathbb{E}x)^2} \to 0$ (used in proof of # triangles)

Corollary: Let x be a r.v. with $\mathbb{E}x > 0$. If $\mathbb{E}(x^2) \leq (\mathbb{E}x)^2(1+o(1))$,

Corollary: Let X be a r.v. with $\mathbb{E}X > 0$. If $\mathbb{E}(X^2) \leq (\mathbb{E}X)^2(1+o(1))$, then $X > 0$ almost surely.