21. Wavelets systems (continued)

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Last time: Necessary conditions on scale function for wavelet basis.

Today:
- Derivation of wavelets from scale function
- Sufficient conditions for orthogonal wavelets

Derivation of wavelets

Let the "mother wavelet" $\Psi(x) = \sum_{k=0}^{d-1} b_k \phi(2x-k)$

We want integer shifts of $\Psi(x)$ to be orthogonal to each other & to scale function $\phi(x)$.

Lemma 11.5 (orthogonality of $\phi(x)$ & $\Psi(x-k)$)

Let $\phi(x) = \sum_{k=0}^{d-1} c_k \phi(2x-k)$ and $\Psi(x) = \sum_{k=0}^{d-1} b_k \phi(2x-k)$

If $\int_{-\infty}^{\infty} \phi(x)\phi(x-k) = \delta(x)$ and $\int_{-\infty}^{\infty} \phi(x)\Psi(x-k)\,dx = 0 \quad \forall k$,

then $\sum_{i=0}^{d-1} c_i b_{i-2k} = 0 \quad \forall k$.

proof.
$$\int_{-\infty}^{\infty} \phi(x)\Psi(x-k)\,dx = \int_{-\infty}^{\infty} \sum_{i=0}^{d-1} c_i \phi(2x-i) \sum_{j=0}^{d-1} b_j \phi(2x-2k-j)\,dx = 0$$

$$\Rightarrow \sum_{i=0}^{d-1}\sum_{j=0}^{d-1} c_i b_j \int_{-\infty}^{\infty} \phi(2x-i)\,\phi(2x-2k-j)\,dx = 0$$

$$\Rightarrow \frac{1}{2}\sum_{i=0}^{d-1}\sum_{j=0}^{d-1} c_i b_j \int_{-\infty}^{\infty} \phi(y)\,\phi(y-2k-j+i)\,dy = 0 \qquad (y = 2x-i)$$

$$\Rightarrow \sum_{i=0}^{d-1}\sum_{j=0}^{d-1} c_i b_j\, \delta(2k+j-i) = 0$$

$$\Rightarrow \sum_{i=0}^{d-1} c_i b_{i-2k} = 0. \qquad \boxed{\diagup}$$

Lemma 11.4 Let $\Psi(x) = \sum_{k=0}^{d-1} b_k \phi(2x-k)$. If $\Psi(x)$ & $\Psi(x-k)$ orthogonal for $k \neq 0$ and

and $\Psi(x)$ has been normalized so

$\int_{-\infty}^{\infty} \Psi(x)\Psi(x-k)\,dx = \delta(k)$, then $\sum_{i=0}^{d-1} b_i b_{i-2k} = 2\,\delta(k)$.

proof Analogous to lemma 11.2.

Computation 11.6 Let $\langle \phi(x), \phi(x-k) \rangle = \delta(k)$

$\langle \phi(x), \psi(x-k) \rangle = 0$ for all k.

Then $b_k = (-1)^k c_{d-1-k}$ is a useful candidate set of coefficients

proof. Let's think of b_k and c_k as discrete functions $b : \mathbb{Z} \to \mathbb{R}$, $c : \mathbb{Z} \to \mathbb{R}$ given by $b[k] = b_k$, $c[k] = c_k$, with support $\{0, 1, \dots, d-1\} \subseteq \mathbb{Z}$.

Recall the discrete convolution $(f * g)[k] = \sum_{m=-\infty}^{\infty} f[m]\, g[k-m]$.

By Lemma 11.5, $\sum_{j=0}^{d-1} c_j b_{j-2k} = 0 \quad \forall k.$

$$\Rightarrow \sum_{j=0}^{\frac{d}{2}-1} c_{2j} b_{2j-2k} + \sum_{j=0}^{\frac{d}{2}-1} c_{2j+1} b_{2j+1-2k} = 0 \quad \forall k$$ (splitting terms of summation)

Let $C_e = (c_0, c_2, \dots, c_{d-2})$ $\quad B_e = (b_0, b_2, \dots, b_{d-2})$

$C_o = (c_1, c_3, \dots, c_{d-1})$ $\quad B_o = (b_1, b_3, \dots, b_{d-1})$

And $C_e^R = (c_{d-2}, \dots, c_2, c_0)$, and so on (reversed).

Then $(C_e * B_e^R)[k] + (C_o * B_o^R)[k] = 0 \quad \forall k.$

By Lemmas 11.2 and 11.4, $\sum_{j=0}^{d-1} c_j c_{j-2k} = 2\delta(k)$ & $\sum_{j=0}^{d-1} b_j b_{j-2k} = 2\delta(k)$.

$$\Rightarrow \sum_{j=0}^{\frac{d}{2}-1} c_{2j} c_{2j-2k} + \sum_{j=0}^{\frac{d}{2}-1} c_{2j+1} c_{2j+1-2k} = 2\delta(k) \quad \forall k$$

$$\sum_{j=0}^{\frac{d}{2}-1} b_{2j} b_{2j-2k} + \sum_{j=0}^{\frac{d}{2}-1} b_{2j+1} b_{2j+1-2k} = 2\delta(k). \quad \forall k$$

$$\Rightarrow (C_e * C_e^R)[k] + (C_o * C_o^R)[k] = 2\delta(k)$$

$$(B_e * B_e^R)[k] + (B_o * B_o^R)[k] = 2\delta(k)$$

Rewrite as matrix $$\left[\begin{pmatrix} C_e & C_o \\ B_e & B_o \end{pmatrix} * \begin{pmatrix} C_e^R & B_e^R \\ C_o^R & B_o^R \end{pmatrix}\right][k] = \begin{pmatrix} 2\delta(k) & 0 \\ 0 & 2\delta(k) \end{pmatrix}$$

Take the discrete Fourier transform

$$\begin{pmatrix} F(C_e) & F(C_o) \\ F(B_e) & F(B_o) \end{pmatrix}\begin{pmatrix} F(C_e^R) & F(B_e^R) \\ F(C_o^R) & F(B_o^R) \end{pmatrix} \underline{[\hat{k}]} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Taking determinant,

$$\left([F(C_e)F(B_o) - F(B_e)F(C_o)][F(C_e^R)F(B_o^R) - F(C_o^R)F(B_e^R)]\right)[\hat{k}] = 4$$

<u>Note</u>: $F(X^R)[\hat{k}] = F(X)^R[\hat{k}]$ time-reversal of Fourier transform

<u>And</u>: $F(X)[\hat{k}]$ is conjugate symmetric about central bin if X is real.

Thus, $F(C_e^R)F(B_o^R) - F(C_o^R)F(B_e^R) = \overline{F(C_e)F(B_o)} - \overline{F(C_o)F(B_e)}$.

$$\Rightarrow \left|F(C_e)F(B_o) - F(B_e)F(C_o)\right|^2 = 4$$

WLOG, $F(C_e)F(B_o) - F(B_e)F(C_o) = 2$. (for this computation; not forced by the condition)

$$\Rightarrow C_e * B_o - C_o * B_e = 2\delta(k).$$

Convolution by C_e^R yields

$$C_e^R * C_e * B_o - C_e^R * C_o * B_e = 2C_e^R * \delta(k)$$

Recall: $-C_e^R * B_e = C_o^R * B_o \Rightarrow C_e^R * C_e * B_o + C_o^R * B_o * C_o = 2C_e^R * \delta(k)$

$$\Rightarrow (C_e^R * C_e + C_o^R * C_o) * B_o = 2C_e^R * \delta(k)$$

$$\Rightarrow 2\delta(k) * B_o = 2C_e^R * \delta(k)$$

$$\Rightarrow C_e^R = B_o.$$

$$\Rightarrow c_i = b_{d-1-i} \text{ for even } i.$$

Similarly, convolution by C_o^R yields $-B_e = C_o^R \Rightarrow c_i = -b_{d-1-i}$ for odd i,

$$\Rightarrow c_i = (-1)^i \cdot b_{d-1-i} \text{ for all } i. \quad \blacksquare$$

<u>Sufficiency condition for orthogonal wavelets.</u>

<u>Lemma 11.7</u> If $b_k = (-1)^k c_{d-1-k}$, then $\int_{-\infty}^{\infty} \phi(x)\psi(2^j x - \ell)\,dx = 0 \quad \forall j, \ell.$

<u>proof</u>. $\int_{-\infty}^{\infty} \phi(x)\psi(x-k)\,dx = \int_{-\infty}^{\infty} \sum_{i=0}^{d-1} c_i \phi(2x-i) \sum_{j=0}^{d-1} b_j \phi(2x-2k-j)\,dx$

proof. $\int_{-\infty}^{\infty}\phi(x)\Psi(x-k)\,dx = \int_{-\infty}^{\infty}\sum_{i=0}^{d-1}c_i\,\phi(2x-i)\sum_{j=0}^{d-1}b_j\,\phi(2x-2k-j)\,dx$

$$= \sum_{i=0}^{d-1}\sum_{j=0}^{d-1}c_i b_j \int_{-\infty}^{\infty}\phi(2x-i)\,\phi(2x-2k-j)\,dx$$

$$= \frac{1}{2}\sum_{i=0}^{d-1}\sum_{j=0}^{d-1}c_i b_j \int_{-\infty}^{\infty}\phi(y)\,\phi(y+i-2k-j)\,dx$$

$$= \frac{1}{2}\sum_i\sum_j(-1)^j c_i c_{d-1-j}\,\delta(i-2k-j) \qquad (i=2k+j)$$

$$= \frac{1}{2}\sum_{j=0}^{d-1}(-1)^j c_{2k+j}\,c_{d-1-j}$$

(because d is even) $\quad = \frac{1}{2}\left[\underline{c_{2k}c_{d-1}} - \underline{c_{2k+1}c_{d-2}} + \cdots + \underline{c_{d-2}c_{2k+1}} - \underline{c_{d-1}c_{2k}}\right] = 0$

Thus, $\int_{-\infty}^{\infty}\phi(2^j x-m)\,\Psi(2^j x-k)\,dx = 0 \quad \forall m, k$ by substitution.

Note that $\phi(x)$ is a linear combination of $\{\phi(2^j x-m)\}_m$.

Thus, $\int_{-\infty}^{\infty}\phi(x)\,\Psi(2^j x-k)\,dx = \int_{-\infty}^{\infty}\sum_m \phi(2^j x-m)\,\Psi(2^j x-k)\,dx = 0.$ $\square$

Lemma 11.8 If $b_k = (-1)^k c_{d-1-k}$, then

$$\int_{-\infty}^{\infty}\frac{1}{2^j}\Psi(2^j x-k)\cdot\frac{1}{2^\ell}\Psi(2^\ell x-m)\,dx = \delta(j-\ell)\,\delta(k-m).$$

proof. $\int_{-\infty}^{\infty}\Psi(x)\Psi(x-k)\,dx = \sum_{i=0}^{d-1}\sum_{j=0}^{d-1}b_i b_j\,\delta(i-2k-j) = \sum_{i=0}^{d-1}b_i b_{i-2k}$

$$= \sum_{i=0}^{d-1}(-1)^i c_{d-1-i}(-1)^{i-2k}c_{d-1-i+2k} = \sum_{i=0}^{d-1}c_{d-1-i}\,c_{d-1-i+2k}$$

Let $j=d-1-i$ $\quad = \sum_{j=0}^{d-1}c_j c_{j+2k} = 2\delta(k)$ ← Lemma 11.2

$$\Rightarrow \int_{-\infty}^{\infty}\Psi(2^\ell x-k)\,\Psi(2^\ell x-m)\,dx = 2^{2\ell}\,\delta(k-m)$$

Also, $\int_{-\infty}^{\infty}\Psi(x)\,\Psi(2^\ell x-k)\,dx = \int_{-\infty}^{\infty}\sum_{i=0}^{d-1}b_i\,\phi(2x-i)\,\Psi(2^\ell x-k)\,dx$

$$= \sum_{i=0}^{d-1}b_i\underbrace{\int_{-\infty}^{\infty}\phi(2x-i)\,\Psi(2^\ell x-k)\,dx}_{=0 \text{ so long as } \ell\ge 1} = 0 \qquad \square$$

$= 0$ so long as $\ell \geq 1$. $\boxed{}$

Thus, we have both necessary & sufficient conditions to generate an orthonormal basis set of wavelets.

$$\phi(x) = \sum_{k=0}^{d-1} c_k \, \phi(2x-k) \qquad \psi(x) = \sum_{k=0}^{d-1} b_k \, \phi(2x-k)$$

$$\sum_{k=0}^{d-1} c_k = 2 \quad \Big\} \text{ 1 linear relation} \qquad b_k = (-1)^k c_{d-1-k}$$

d is even

$$\underbrace{\sum_{k=0}^{d-1} c_i \, c_{i-2k} = 2\,\delta(k)}_{d/2 \text{ quadratic eq}}$$

Then, we start with d degrees of freedom in the c_k's, but they have to satisfy $\frac{d}{2}+1$ relations

$\Rightarrow$ We have $\frac{d}{2} - 1$ df left to design wavelet system for properties of interest.

Wavelet transform

If $f(x) = \sum_{k=0}^{\infty} a_{jk} \, \phi_{jk}(x)$, where $\phi_{jk}(x) = \phi(2^j x - k)$

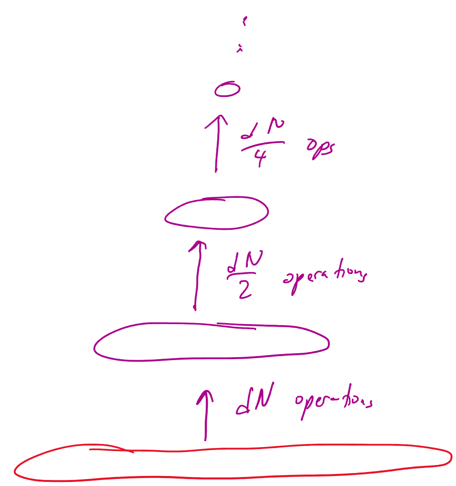
then $a_{jk} = \int_{-\infty}^{\infty} f(x)\, \phi_{jk}(x)$ by orthogonality.

$$= \int_{-\infty}^{\infty} f(x) \sum_{m=0}^{d-1} c_m \, \phi_{j+1,\,2k+m}(x)\, dx$$

$$= \sum_{m=0}^{d-1} c_m \int_{-\infty}^{\infty} f(x)\, \phi_{j+1,\,2k+m}(x)\, dx$$

$$= \sum_{m=0}^{d-1} c_m \, a_{j+1,\,2k+m}$$

$\frac{N}{2}$ samples, level $j-1$



Let $n = 2k+m$, so $m = n - 2k$

$$\Rightarrow a_{jk} = \sum_{n=2k}^{2k+d-1} c_{n-2k} \, a_{j+1,\,n}$$

N samples, level j

This gives us a formula for moving up the tree to compute

Total: $2dN$ operations
$= O(N)$ since d

$n=2k$

This gives us a formula for moving up the tree to compute coefficients of the scale function from higher res. samples

But, of course, using that we can compute coefficients in the wavelet basis.

Total: $2dN$ operations $= O(N)$ since d is constant for the wavelet.

Also local so can be computed on streaming basis