14. Erdos-Renyi graphs and phase transitions

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Erdös-Renyi random graphs

Let $G(n,p)$ be a graph-valued random variable, with vertices V and edges E.

- $n = |V|$
- $p = \text{Prob}((v_i, v_j) \in E)$ for any $i \neq j$. (prob. of each edge is independent)

We will use this model to study certain types of phase transition behavior

Thm 8.1 Let $v_1 \in V$ of the random graph $G(n,p)$. Aside: $\mathbb{E}\deg(v_i) = p(n-1)$
Let $\alpha \in (0, \sqrt{np})$. Then

$$\text{Prob}\left(|(n-1)p - \deg(v_1)| \geq \alpha\sqrt{(n-1)p}\right) \leq 3e^{-\frac{\alpha^2}{8}}$$

proof. $\deg(v_1) = \sum_{i=2}^{n} \mathbb{1}_{1,i}$, where $\mathbb{1}_{i,j} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{else} \end{cases}$

Then the proof follows from Chernoff bounds. (Thm 12.6)

$$\text{Prob}\left(|\deg(v_1) - (n-1)p| \geq c(n-1)p\right) \leq 3e^{-(n-1)pc^2/8}$$

Let $c = \frac{\alpha}{\sqrt{(n-1)p}}$.

$\blacksquare$

Corollary 8.2 Suppose $\varepsilon > 0$. If $p \geq \frac{9\ln n}{(n-1)\varepsilon^2}$, then with $1 - o(1)$ probability,
$\deg(v_i) \in [(1-\varepsilon)(n-1)p, (1+\varepsilon)(n-1)p] \ \forall i$.

proof. Let $\alpha = \varepsilon\sqrt{(n-1)p}$ in Thm 8.1

Then for a given i, failure prob is $\leq 3e^{-\varepsilon^2(n-1)p/8}$

By union bound, failure prob for any i $\leq 3ne^{-\varepsilon^2(n-1)p/8}$

$p \geq \frac{9\ln n}{(n-1)\varepsilon^2}$ $\qquad \leq 3ne^{\frac{-9\ln n}{8}} = 3n \cdot n^{-\frac{9}{8}} = 3n^{-\frac{1}{8}} = o(1)$.

$\blacksquare$

i.e. if $\underbrace{p = \Omega\left(\frac{\log n}{n}\right)}$, then with vanishing prob., all vertices have tightly constrained degree.

Note $p = \Omega\left(\frac{1}{n}\right)$ fails. Look at prob. degree 0 when $p = \frac{1}{n}$.

Claim: $G(n, \frac{d}{n})$ has in expectation $\approx \frac{d^3}{6}$ triangles.

Moral justification of independence from n:

As n increases # triples grows with n^3

But each pair has $\frac{d}{n}$ prob of having an edge, so 3 pairs $= \frac{d^3}{n^3}$ chance

$\frac{d^3}{n^3}$ chance balances out n^3 # triples

proof. Let Δ_{ijk} be the indicator variable for existence of triangle v_i, v_j, v_k.

$$\text{Then } \mathbb{E}(\#\text{ triangles}) = \mathbb{E}\Big(\sum_{ijk} \Delta_{ijk}\Big) = \sum_{ijk} \mathbb{E}(\Delta_{ijk}) = \binom{n}{3}\left(\frac{d}{n}\right)^3 = \underbrace{\frac{n(n-1)(n-2)}{6}} \cdot \frac{d^3}{n^3} \approx \frac{d^3}{6}$$

linearity of expectation doesn't depend on independence

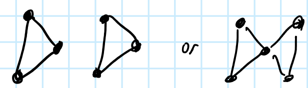
But this only talks about the expected # triangles. Let's try to show that with prob bounded away from 0, $\exists$ at least one triangle in $G(n, \frac{d}{n})$

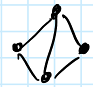
i.e. rule out occasional all triangle graphs and occasional 0-triangle graphs.

Let $x = \#$ triangles, $x = \sum_{ijk} \Delta_{ijk}$. We will use a 2nd moment argument.

$$\mathbb{E}(x^2) = \mathbb{E}\Big(\sum_{ijk} \Delta_{ijk}\Big)^2 = \mathbb{E}\Big(\sum_{\substack{i,j,k \\ i',j',k'}} \Delta_{ijk}\, \Delta_{i'j'k'}\Big)$$

Split the sum into 3 parts.

$S_1 = \{i, j, k, i', j', k' \mid \Delta_{ijk}$ and $\Delta_{i'j'k'}$ share no edges$\}$ 

$S_2 = \{i, j, k, i', j', k' \mid \Delta_{ijk}$ and $\Delta_{i'j'k'}$ share exactly 1 edge$\}$ 

$S_3 = \{i, j, k, i', j', k' \mid \Delta_{ijk} = \Delta_{i'j'k'}\}$ 

$$\mathbb{E}\Big(\sum_{S_1} \Delta_{ijk}\, \Delta_{i'j'k'}\Big) = \sum_{S_1} \underbrace{\mathbb{E}(\Delta_{ijk})\, \mathbb{E}(\Delta_{i'j'k'})}_{\text{ind. because no edges shared}} \leq \Big(\sum_{\substack{\text{all} \\ i,j,k}} \mathbb{E}(\Delta_{ijk})\Big)\Big(\sum_{\substack{\text{all} \\ i',j',k'}} \mathbb{E}(\Delta_{i'j'k'})\Big) = (\mathbb{E}x)^2$$

$$\mathbb{E}\Big(\sum_{S_2} \Delta_{ijk}\, \Delta_{i'j'k'}\Big) = \underbrace{\binom{n}{4}}_{\text{ways to choose 4 vertices}} \underbrace{\binom{4}{2}}_{\text{ways to choose 2 vertices lacking an edge}} \underbrace{p^5}_{\text{chance remaining 5 edges are present}} \approx \frac{n^4}{24} \cdot 6 \cdot p^5 = \frac{1}{4} n^4 p^5 = \frac{1}{4} n^4 \cdot \frac{d^5}{n^5} = \frac{1}{4} \cdot \frac{d^5}{n} = o(1)$$

$$\mathbb{E}\Big(\sum_{S_3} \Delta_{ijk}\, \Delta_{i'j'k'}\Big) = \mathbb{E}\Big(\sum_{S_3} \Delta_{ijk}\Big) = \mathbb{E}x$$

$$\Rightarrow \mathbb{E}(x^2) \leq (\mathbb{E}x)^2 + \mathbb{E}x + o(1)$$

$$\Rightarrow Var(x) = \mathbb{E}(x^2) - (\mathbb{E}x)^2 \leq \mathbb{E}x + o(1)$$

Then $Prob(x=0) \leq Prob(|x - \mathbb{E}x| \geq \mathbb{E}x)$

By Chebychev, $Prob(x=0) \leq \frac{Var(x)}{(\mathbb{E}x)^2} \leq \frac{\mathbb{E}x + o(1)}{(\mathbb{E}x)^2} \leq \frac{6}{d^3} + o(1)$

Thus, if $d > \sqrt[3]{6} \approx 1.8$, $Prob(x=0) < 1$, so $G(n, \frac{d}{n})$ has a triangle with non-zero probability.

For $d < \sqrt[3]{6}$, $\mathbb{E}x = \frac{d^3}{6} < 1$, so not many triangles to go around

Intuitively, need many vertices with $\deg \geq 2$ to have triangles.

Phase transitions

Def. If $\exists\, p(n)$ s.t. when $\lim_{n\to\infty} \frac{p_1(n)}{p(n)} = 0$, $G(n, p_1(n))$ lacks a property (almost surely),

when $\lim_{n\to\infty} \frac{p_2(n)}{p(n)} = \infty$, $G(n, p_2(n))$ has a property (almost surely),

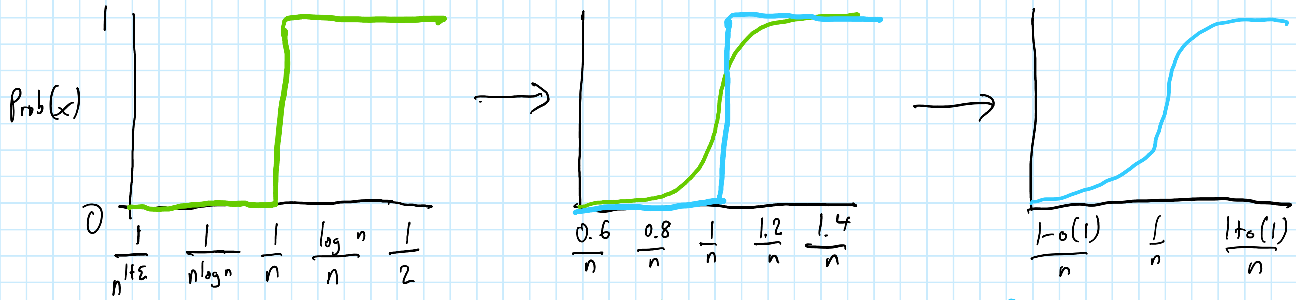
then a phase transition for the property occurs at threshold $p(n)$.

Def. If for $cp(n)$, $c < 1$, $G(n, cp(n))$ lacks a prop. almost surely,

$c > 1$, $G(n, cp(n))$ has a prop. almost surely,

then $p(n)$ is a sharp threshold.

As the average degree increases in an Erdős-Renyi graph, structural properties suddenly change.



asymp phase trans at $\frac{1}{n}$

may be smoother when zoomed in unless sharp

If prev. sharp, can zoom in even more

Phase transitions of Erdős–Renyi graph

Probability	Behavior
$p = o\left(\frac{1}{n}\right)$	Forest of trees, component size $O(\log n)$
$p = \frac{d}{n}$, $d < 1$	Some cycles, component size $O(\log n)$
$p = \frac{d}{n}$, $d = 1$	Components of size $O(n^{2/3})$
$p = \frac{d}{n}$, $d > 1$	Giant component + $O(\log n)$ components
$p = \frac{1}{2} \cdot \frac{\ln n}{n}$	Giant component + isolated vertices
$p = \frac{\ln n}{n}$	No isolated vertices. Appearance of Hamiltonian circuit. Diameter $O(\log n)$
$p = \sqrt{\frac{2 \ln n}{n}}$	Diameter 2
$p = \frac{1}{2}$	Clique of size $(2-\varepsilon)\ln n$.

How to prove these properties? Use so-called moment methods.

First-moment method

Let $x(n)$ denote the number of occurences of an item in a random graph. If $\mathbb{E}x(n) \to 0$ as $n \to \infty$, then a random graph almost surely has no occurences of the item.

proof. Markov's inequality, x is non-negative.

$$\mathrm{Prob}(x \geq a) \leq \frac{1}{a}\mathbb{E}x, \text{ so } \mathrm{Prob}(x(n) \geq 1) \leq \mathbb{E}x(n) \to 0 \quad \text{as } n \to \infty.$$

Second-moment method

Recall $f(x) = o(g(x))$ iff $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$

Let $x(n)$ be a random variable with $\mathbb{E}x > 0$. If $\mathrm{Var}(x) = o((\mathbb{E}x)^2)$, then $x > 0$ almost surely.

proof. $\mathrm{Prob}(x \leq 0) \leq \mathrm{Prob}(|x - \mathbb{E}x| \geq \mathbb{E}x)$

By Chebyshev, $\leq \frac{\mathrm{Var}(x)}{(\mathbb{E}x)^2} \to 0.$

(used in proof of # triangles)

Corollary Let x be a r.v. with $\mathbb{E}x > 0$. If $\mathbb{E}(x^2) \leq (\mathbb{E}x)^2(1 + o(1))$,

Corollary Let X be a r.v. with $\mathbb{E}X > 0$. If $\mathbb{E}(X^2) \leq (\mathbb{E}X)^2 (1 + o(1))$,

then $X > 0$ almost surely.

Harder to use 2nd moment method because it can be hard to compute variance without independence (i.e. $\mathbb{E}XY \neq \mathbb{E}X\,\mathbb{E}Y$)

In looking for a phase transition, almost always the transition in probability of an item occurring occurs when the expected number of items transitions