15. Erdos-Renyi diameter and isolated vertices

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Thm: The property that $G(n,p)$ has diameter 2 has a sharp threshold at $p = \sqrt{2} \cdot \sqrt{\frac{\ln n}{n}}$.

(i.e. If $p = c\sqrt{\frac{\ln n}{n}}$, for $c < \sqrt{2}$, diameter > 2 a.s.
$c > \sqrt{2}$, diameter ≤ 2 a.s.)

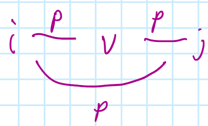
proof. If diameter > 2, then $\exists$ non-adjacent vertices i and j s.t. no other vertex is adj to both i and j. Call such a pair "bad."

Let indicator $I_{ij} = 1$ iff pair (i,j) is bad.

Let $x = \sum_{i<j} I_{ij}$, # of bad vertices.

$$\mathbb{E}x = \underbrace{\binom{n}{2}}_{\substack{\#\text{ pairs}\\ i<j}} \underbrace{(1-p)}_{\substack{\text{prob } \nexists (i,j)\\ \text{so non-adj}}} \underbrace{(1-p^2)}_{\substack{\text{prob a vertex}\\ \text{is not adj}\\ \text{to both}\\ i \,\&\, j}}^{n-2}$$

$n-2$ — # of other vertices



Setting $p = c\sqrt{\frac{\ln n}{n}}$, $\mathbb{E}x \approx \frac{n^2}{2}\left(1 - c\sqrt{\frac{\ln n}{n}}\right)\left(1 - c^2 \frac{\ln n}{n}\right)^n$

$$\approx \frac{n^2}{2} \exp\left(-c^2 \cdot \ln n\right) \approx \frac{1}{2} n^{2-c^2}$$

For $c > \sqrt{2}$, $\lim_{n\to\infty} \mathbb{E}x = 0$. By the 1st moment method, $G(n,p)$ a.s. has no bad pair, and hence diameter $\leq$ 2.

Now consider $c < \sqrt{2}$, where $\lim_{n\to\infty} \mathbb{E}x = \infty$. We will use 2nd moment method.

$$\mathbb{E}(x^2) = \mathbb{E}\left(\sum_{i<j} I_{ij}\right)^2 = \mathbb{E}\left(\sum_{i<j} I_{ij} \sum_{k<l} I_{kl}\right) = \mathbb{E}\left(\sum_{\substack{i<j\\k<l}} I_{ij} I_{kl}\right) = \sum_{\substack{i<j\\k<l}} \mathbb{E}(I_{ij} I_{kl})$$

Use the same idea with triangles to partition the set.

$$= \sum_{\substack{i<j\\k<l}} \mathbb{E}(I_{ij} I_{kl}) + \sum_{\substack{\{i,j,k\}\\ i<j}} \mathbb{E}(I_{ij} I_{ik}) + \sum_{i<j} \mathbb{E}(I_{ij}^2)$$

all i,j,k,l distinct   all i,j,k distinct

When all 4 vertices are distinct, must be two bad pairs (i,j) and (k,ℓ) for $I_{ij} I_{k\ell} = 1$. Then $\forall u \notin \{i,j,k,\ell\}$, at least one of $(i,u), (j,u)$ is absent
" " " " $(k,u), (\ell,u)$ is absent.

The probability of both absences is $(1-p^2)^2$

So $\mathbb{E}(I_{ij} I_{k\ell}) \le (1-p^2)^{2(n-4)} \le \left(1 - c^2 \frac{\ln n}{n}\right)^{2n} (1+o(1)) \le n^{-2c^2}(1+o(1))$

$$\Rightarrow \sum_{\substack{i<j \\ k<\ell \\ \text{all distinct}}} \mathbb{E}(I_{ij} I_{k\ell}) \le \frac{1}{4} n^{4-2c^2}(1+o(1))$$ (because $< \frac{1}{4}$ of n^4 4-tuples have $i<j$, $j<k$)

When only 3 distinct vertices, if $I_{ij} I_{ik} = 1$, then $\forall u \notin \{i,j,k\}$, either

there is no edge (i,u),

or there is an edge (i,u) and both (j,u), (k,u) are absent.

The prob. is $1 - p + p(1-p)^2 = 1 - 2p^2 + p^3 \approx 1 - 2p^2$ (for one u)

Thus, $\mathbb{E}(I_{ij} I_{ik}) \approx (1-2p^2)^{n-3} \approx \exp(-2p^2(n-3)) \approx \exp(-2c^2 \ln n) \approx n^{-2c^2}$

$$\Rightarrow \sum_{\substack{\{i,j,k\} \\ i<j}} \mathbb{E}(I_{ij} I_{ik}) \le n^3 \cdot n^{-2c^2} = n^{3-2c^2}.$$

When only 2 distinct vertices $\sum_{i,j} \mathbb{E}(I_{ij})^2 = \mathbb{E}x \approx \frac{1}{2} n^{2-c^2}$.

Together $\mathbb{E}(x^2) \le \frac{1}{4} n^{4-2c^2} + n^{3-2c^2} + \frac{1}{2} n^{2-c^2} = \frac{1}{4} n^{4-2c^2}(1 + 4n^{-1} + 2n^{c^2-2})$

If $c < \sqrt{2}$, $\mathbb{E}(x^2) \le (\mathbb{E}x)^2(1+o(1))$

By a 2nd moment argument, the graph a.s. has at least one bad pair, so the diamenter > 2. $\square$

Thm The disappearance of isolated vertices in $G(n,p)$ has a sharp threshold of $\frac{\ln n}{n}$.

proof. let $x = \#$ isolated vertices.

Then $\mathbb{E}x = n(1-p)^{n-1}$

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Let $p = c\frac{\ln n}{n}$. Then $\lim_{n\to\infty} \mathbb{E}x = \lim_{x\to\infty} n\left(1 - \frac{c\ln n}{n}\right)^n = \lim_{n\to\infty} n e^{-c\ln n} = \lim_{n\to\infty} n^{1-c}$

If $c>1$, $\mathbb{E}x \to 0$, so by 1st moment argument, almost all graphs have isolated vertices.

If $c<1$, $\mathbb{E}x\to\infty$, so need a 2nd moment argument.

Assume $c<1$. Let $x = \sum_i I_i$, where indicator $I_i = \begin{cases} 1 & \text{if } i \text{ is isolated} \\ 0 & \text{else} \end{cases}$

$$\text{Then } \mathbb{E}(x^2) = \sum_{i=1}^n \mathbb{E}(I_i^2) + 2\underbrace{\sum_{i<j} \mathbb{E}(I_i I_j)}_{\text{isolation is not independent}}$$

$$= \mathbb{E}x + n(n-1)\,\mathbb{E}(I_1 I_2)$$
$$= \mathbb{E}x + n(n-1)(1-p)^{n-1}(1-p)^{n-2}$$
$$= \mathbb{E}x + n(n-1)(1-p)^{2(n-1)-1}$$

$$\text{Thus, } \frac{\mathbb{E}(x^2)}{(\mathbb{E}x)^2} = \frac{n(1-p)^{n-1} + n(n-1)(1-p)^{2(n-1)-1}}{n^2(1-p)^{2(n-1)}} = \frac{1}{n(1-p)^{n-1}} + \left(1-\frac{1}{n}\right)\frac{1}{1-p}$$

For $p = c\frac{\ln n}{n}$ with $c<1$, $\lim_{n\to\infty}\mathbb{E}x = \infty$ and

$$\lim_{n\to\infty}\frac{\mathbb{E}(x^2)}{(\mathbb{E}x)^2} = \lim_{n\to\infty}\left[\underbrace{\frac{1}{n^{1-c}}}_{\to 0} + \left(1-\frac{1}{n}\right)\cdot\frac{1}{1-c\frac{\ln n}{n}}\right] = \lim_{n\to\infty}\left(\frac{1}{1-c\frac{\ln n}{n}}\right) = 1 + o(1).$$

By 2nd moment argument, almost all graphs have isolated vertices for $c<1$.