17. Percolation theory

Tuesday, October 19, 2021 11:36 AM

We have spent basically all of our time on random graphs where each vector was free to connect to every other vector with probability p.

But this is not very realistic for the physical world, where there may be geometric constraints. Instead, let's consider conductivity through a material

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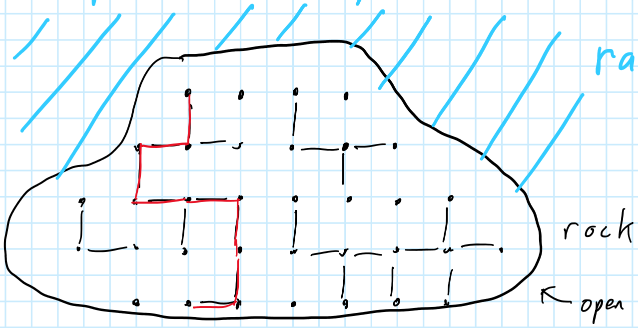
The term percolation theory comes from open/closed channels for a fluid to flow



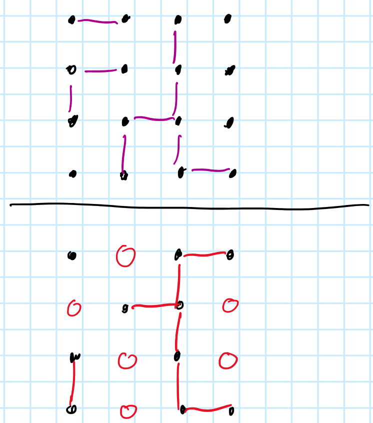
Is there an open path from top to bottom? i.e. is the rock porous?

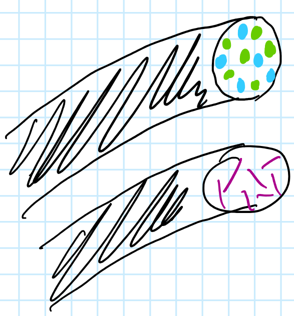
- Can approximate lattice as infinite if rock much bigger than channels

Consider a square lattice $\mathbb{Z}^2$

Bond percolation: all vertices present, but (open) edges present w.p. p.

Site percolation: vertices present w.p. p, all edges present b/t neighbors





Percolation in composite wires = electrical conductivity.

Many polymers are insulating, but you can get the strength of a polymer without needing it to be homogeneous, so can dope it with a conductive filler.

Site percolation: If displacement on 1-1 basis.

Bond percolation: If insulator only breaks bonds by fitting sometimes in between conductive atoms.

Basic questions:

- Does there exist an infinite open cluster? ⟵ edge percolation

- Does there exist an infinite open cluster? ⟸ edge percolation
- What is the size distribution of open clusters?

Relationship to network-based epidemic models, but those are not in vogue right now.

Let our 2D edge percolation graph have vertices $\mathbb{Z}^2$ and edges b/t all adj. vertices with distance 1 (i.e. vertically and horizontally).

Let each edge be "open" w.p. p, and "closed" w.p. $1-p$.

Def. Let $C(x)$ denote the component containing x in a 2D edge percolation graph.
↑ random variable dependent on presence of open edges.

Def. Let $C = C(0)$. Define $\theta(p) = \text{Prob}_p(|C| = \infty)$, the probability that the origin is in an infinite component.

Def. Let p_c be a constant, such that for $p < p_c$, $\theta(p) = 0$, and for $p > p_c$, $\theta(p) > 0$.

Aside: Is there $p < 1$, such that $\theta(p) = 1$?

Aside: By Kolmogorov's 0-1 law (stating tail events have either 0 or 1 prob.), if $\theta(p) > 0$, then $\exists$ a.s. an infinite open component.

Theorem: If $p < \frac{1}{3}$, $\theta(p) = 0$.

proof. Use 1st moment method.

Let F_n = event $\exists$ self-avoiding open path of length n, starting at 0.

For any given self-avoiding path in $\mathbb{Z}^2$, prob. all edges open $= p^n$.

Total # self-avoiding paths of length n $\leq 4(3^{n-1})$
(because we can't backtrack after 1st step)

$\Rightarrow \text{Prob}(F_n) \leq 4(3^{n-1})p^n \longrightarrow 0$ as $n \to \infty$ since $p < \frac{1}{3}$.

$\{|C| = \infty\} \subseteq F_n \quad \forall n$, (because if you have an infinite component, can find a self-avoidant path of length n)

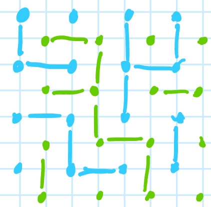
$\Rightarrow \text{Prob}\{|C| = \infty\} = 0 \Rightarrow \theta(p) = 0.$

$\Rightarrow p_c \geq \frac{1}{3}.$ $\boxtimes$

Theorem (Harris, 1960) $\theta(\tfrac{1}{2}) = 0$. $(p_c \geq \tfrac{1}{2})$

Outline: Use self-duality of $\mathbb{Z}^2$.

We can define a dual graph by translating down and to the right by $(0.5, 0.5)$, and defining an edge to be present in the dual when it does not cross an edge in the original graph.



Note: Edge prob p in original $\Rightarrow$ Edge prob $1-p$ in dual.

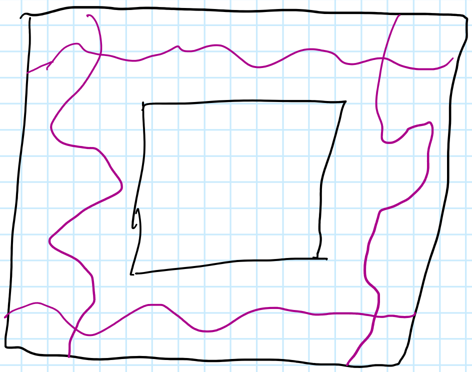
Suppose $p < p_c$. Then $C(0)$ is finite a.s.

$\Rightarrow$ exists an open cycle in dual graph encircling 0.

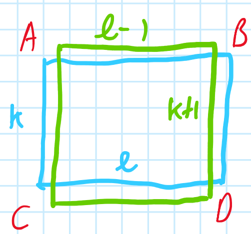
Conversely, if $\exists$ open cycle in dual graph encircling $C(0)$ is finite because trapped.

So, just need to show existence of open cycle around 0 to prove $\theta(p) = 0$.

Let $p = \tfrac{1}{2}$. Let's consider prob of an open cycle in an annulus composed of 4 separate open paths across rectangles.

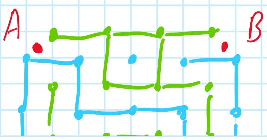


Lemma: Let R be a rectangular $k \times \ell$ portion of the original lattice, and R' be the corresponding rectangular $(k+1) \times (\ell - 1)$ portion of the dual lattice.

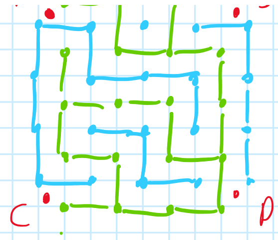


Then the exists either an open horizontal path in R or an open vertical path in R'.

proof. Almost entirely geometric.



- Walk between "walls" constructed by the open edges of original and dual graphs.
- Start at A, and keep original wall on right and dual wall on left



- Start at A, and keep original wall on right and dual wall on left
- Then walk must end at B or C because original wall on right and dual on left.
  Can't end at D since walls on wrong side.
- Walk can't end in middle because would require dual and original edges to intersect.

Walk is simultaneously a walk on both R and R'.
End at B = horizontal walk on R
End at C = vertical walk on R'.

Let $\underset{p}{\text{Prob}}(H(R))$ be prob of horizontal path in R
$\underset{1-p}{\text{Prob}}(V(R'))$ be prob. of horizontal path in R'

$$\Rightarrow \underset{p}{\text{Prob}}(H(R)) + \underset{1-p}{\text{Prob}}(V(R')) = 1$$

$$\Rightarrow \underset{\frac{1}{2}}{\text{Prob}}(H(R)) + \underset{\frac{1}{2}}{\text{Prob}}(V(R')) = 1.$$

If R is an $n \times (n+1)$ rectangle, then R' is an $(n+1) \times n$ rectangle.

$$\Rightarrow \underset{\frac{1}{2}}{\text{Prob}}(H(R)) = \underset{\frac{1}{2}}{\text{Prob}}(V(R')) = \frac{1}{2}.$$

Let S be an $n \times n$ square. Horizontal distance to travel is $< n+1$ (in R),

so $\underset{\frac{1}{2}}{\text{Prob}}(H(S)) \geq \frac{1}{2}$, $\forall n$.

We will now use a simplification of an argument by Russo, Seymour, Welsh (RSW theory) to prove there exists with positive prob. independent of n an open cycle in the annulus.

Thm RSW [Russo-Seymour-Welsh]:

Let $H_{n,kn} = H(R)$, where R is an $n \times kn$ rectangle.

For all k, $\exists c_k$ s.t. $\forall n$, $\underset{\frac{1}{2}}{\text{Prob}}(H_{n,kn}) \geq c_k$.

Definition: Let the state space $\Omega := \{0,1\}^J$. There is a partial order on Ω given by $w \preceq w'$ if $w_i \leq w_i'$ for all $i \in J$.

A function $f : \Omega \to \mathbb{R}$ is increasing if $\omega \preceq \omega'$ implies that $f(\omega) \le f(\omega')$

An event is increasing if its indicator function is increasing.

Note: If J is a set of edges in a graph and x & y are vertices, then the event that there is an open path is an increasing event.

Thm 6.2: Let $X := \{X_i\}_{i \in J}$ be independent r.v. taking values 0 and 1. Let f and g be increasing functions. Then

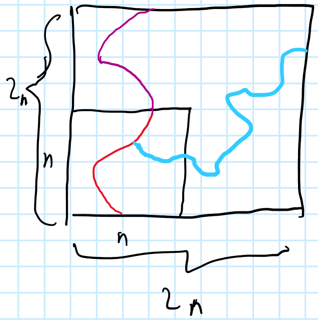
$$\mathbb{E}\left(f(X)g(X)\right) \ge \mathbb{E}f(x) \cdot \mathbb{E}g(x)$$

proof: See reference Steif, 2009. Proof by induction & direct computation of expectations.

Intuitively, while two events like existence of vertical and horizontal paths are not independent, they are positively correlated because more edges are good for both events.

proof of RSW (not tight)

Let $F_1 = \left\{ \begin{array}{l} \text{open path connects right side of } 2n \times 2n \text{ square with} \\ \text{top and bottom of } n \times n \text{ square in lower left quadrant} \end{array} \right\}$ ← event



$\text{Prob}_{\frac{1}{2}}(J_{2n,2n}) \ge \frac{1}{2}$, $\text{Prob}_{\frac{1}{2}}(J_{n,n}) \ge \frac{1}{2}$ ← crossing path.

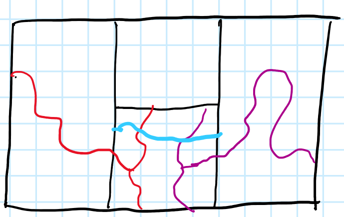
Let g be an open vertical path in lower left $n \times n$ square, and g' be its reflection about $y = n$

Let h be a horizontal path in the $2n$ by $2n$ square. By symmetry, must cross either g or g' with same prob of either.

$$\text{So} \quad \text{Prob}_{\frac{1}{2}}(F_1) \ge \frac{1}{2}\text{Prob}_{\frac{1}{2}}(J_{2n,2n})\,\text{Prob}_{\frac{1}{2}}(J_{n,n}) = 2^{-3}$$

(Thm 6.2)

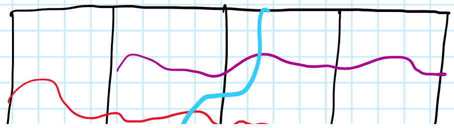
Let event $F_2 = \{$ open path across $2n \times 3n$ rectangle.



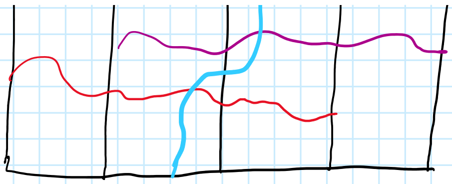
We can break up into two instances of F_1, coupled with an open path in middle lower $n \times n$ square.

$$\text{So} \quad \text{Prob}(F_2) \ge \left(\text{Prob}(F_1)\right)^2 \cdot \text{Prob}(J_{n \times n}) \ge 2^{-7}.$$

Let event $F_3 = \{$ open path across $4n \times 2n$ rectangle $\}$



$$\text{Prob}(F_3) \ge \left(\text{Prob}(F_2)\right)^2 \cdot \text{Prob}(J_{2n \times 2n}) \ge 2^{-15}.$$



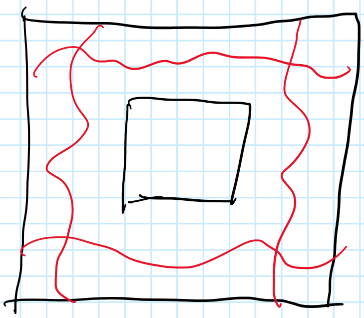
$\text{Prob}(F_3) \geq \left(\text{Prob}(F_2)\right)^2 \cdot \text{Prob}(J_{2n \times 2n}) \geq 2^{-15}$.

No actual dependence on n, so $\text{Prob}(F_3) = \text{Prob}(J_{4n \times 2n}) = \text{Prob}(J_{2n \times n})$.

Continue process to get $\text{Prob}_{\frac{1}{2}}(J_{kn \times n}) = c_k$ for some constant $c_k > 0$.

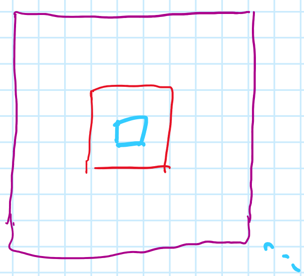
In particular, $\text{Prob}_{\frac{1}{2}}(J_{3n \times n}) \geq 2^{-31}$.

Let's go back to rectangular annulus, let $F_A = \left\{ \begin{array}{l} \text{open cycle in } 3n \times 3n \text{ annulus} \\ \text{not including center } n \times n \text{ box} \end{array} \right\}$



Then $\text{Prob}(F_A) \geq (2^{-31})^4 = 2^{-124}$, ind. of n.

Key: Surround the annulus with infinitely many larger annuli, each with ind. prob. of having an open cycle.



$\text{Prob}(\text{no open cycle in annulus}) \leq 1 - 2^{-124}$.

$\text{Prob}(\text{no open cycle in } k \text{ non-overlapping annuli}) \leq (1 - 2^{-124})^k$

As $k \to \infty$, prob. $\to 0$, so a.s. there is an open cycle surrounding the origin.

$\Rightarrow \theta\left(\frac{1}{2}\right) = 0 \quad \Rightarrow \quad p_c \leq \frac{1}{2}$.

Thm. [Kesten, 1982] $p_c \geq \frac{1}{2}$.

Not proved here as proof quite long.