18. Wavelets

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Wavelets

Want an orthonormal basis of the vector space of functions.

Ex. Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i \left(\frac{n}{T}\right) x} \quad \text{for} \quad x \in \left[-\frac{T}{2}, \frac{T}{2}\right]$$

Fourier transform

x_i

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \, dx$$

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} \, d\xi$$

But sines + cosines are distributed in support, so want something with finite support that's also efficiently computable

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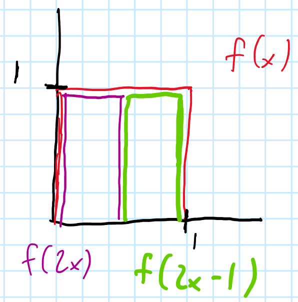
Define: A **dilation** is a mapping that scales all distances by the same factor.

A **dilation equation** is an equation where a function is defined in terms of shifted, scaled versions of itself.

Ex. $f(x) = \sum_{k=0}^{d-1} c_k f(2x-k)$

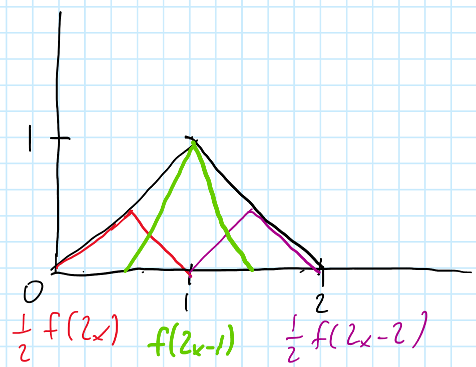
Ex. $f(x) = f(2x) + f(2x-1)$

One sol: $f(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & \text{else} \end{cases}$



Ex. $f(x) = \frac{1}{2} f(2x) + f(2x-1) + \frac{1}{2} f(2x-2)$

One sol: $f(x) = \begin{cases} x, & 0 \le x < 1 \\ -x+1, & 1 \le x < 2 \\ 0, & \text{else} \end{cases}$



If a dilation eq is of form $\sum_{k=0}^{d-1} c_k f(2x-k)$, then we say that all dilations in the [illegible]

If a dilation eq is of form $\sum_{k=0} c_k f(2x-k)$, then we say that all dilations in the eq. are factor 2 reductions.

Lemma 11.1 Consider a dilation eq $f(x) = \sum_{k=0}^{d-1} c_k f(2x-k)$ where all dilations are factor 2 reductions.

Then either: $\sum_{k=0}^{d-1} c_k = 2$ or $\int_{-\infty}^{\infty} f(x)\,dx = 0$.

solution.

proof.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \sum_{k=0}^{d-1} c_k f(2x-k)\,dx$$

$$= \sum_{k=0}^{d-1} \int_{-\infty}^{\infty} c_k f(2x-k)\,dx$$

(allowed if 1-norm of function is finite)

$$= \sum_{k=0}^{d-1} c_k \int_{-\infty}^{\infty} f(2x)\,dx$$

(because integrating over entire real line, so shifts irrelevant)

$$= \frac{1}{2} \sum_{k=0}^{d-1} c_k \int_{-\infty}^{\infty} f(x)\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 0 \quad \text{or} \quad \sum_{k=0}^{d-1} c_k = 2$$

both allowed and give nonzero sols.

Haar wavelet

Let scale function $\phi(x)$ be a sol to $f(x) = f(2x) + f(2x-1)$ e.g. $\phi(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & \text{else} \end{cases}$

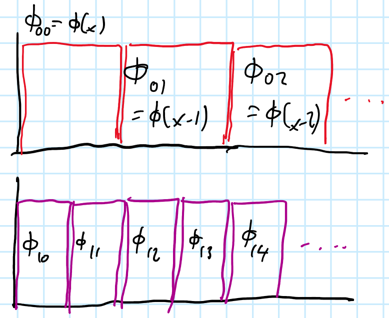
Let $\phi_{jk}(x) = 2^{\frac{j}{2}} \phi(2^j x - k)$

needed later for orthonormal basis, but we are going to ignore it for ease of notation.

Let $V_j = \operatorname{span}\{\phi_{jk} \mid k \in \mathbb{N}\}$

$V_0 = \operatorname{span}\{\phi_{00}, \phi_{01}, \phi_{02}, \dots\}$

$V_1 = \operatorname{span}\{\phi_{10}, \phi_{11}, \phi_{12}, \dots\}$



Note $V_0 \subseteq V_1 \subseteq V_2 \subseteq \dots \subseteq V_j \subseteq V_{j+1} \subseteq \dots$ ← This fact is guaranteed because ϕ is a dilation equation.

Recall: We can define orthogonality w.r.t. any inner product.

Let $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) g(x)\, dx$.

Then f and g are orthogonal $\iff \langle f, g \rangle = 0$

orthonormal $\iff \langle f, g \rangle = 0, \ \langle f, f \rangle = 1, \ \langle g, g \rangle = 1.$

Note: $\langle \phi_{jk}, \phi_{j\ell} \rangle = 0 \quad \forall j \neq \ell$ because of nonoverlapping supports.

So $\{\phi_{j0}, \phi_{j1}, \ldots\}$ is a basis of V_j.

$\mathrm{span}\{\phi_{jk}\}_{j,k}$ = space of functions, but is not a Schauder (countable) basis because not linear. ind.

We want to construct a basis out of these functions

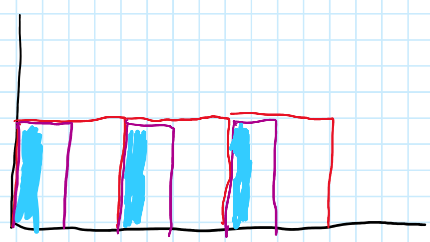
But $\phi_{jk} = \phi_{j+1,2k} + \phi_{j+1,2k+1}$

We could in theory delete ϕ_{jk}, but that doesn't work, since we are left with only "$\phi_{\infty,k}$"

Delete $\phi_{j+1,2k+1}$ from our set of functions, $\forall k$.

i.e. we are removing all $\phi_{j,k}$, where $j > 0$ and k is odd.

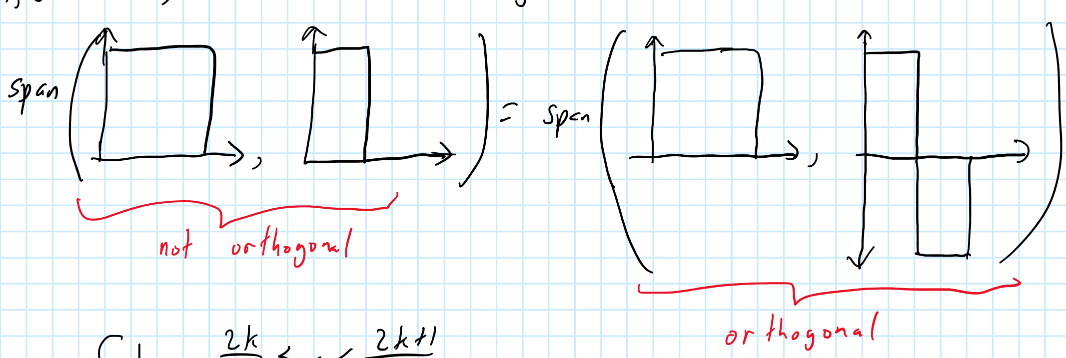
Then we get
$$\{\phi_{00}, \phi_{01}, \phi_{02}, \ldots$$
$$\phi_{10}, \phi_{12}, \phi_{14}, \ldots$$
$$\phi_{20}, \phi_{22}, \phi_{24}, \ldots \}$$
$$\vdots$$



which form a basis for our set of functions.

But $\langle \phi_{jk}, \phi_{j+1,2k} \rangle \neq 0$, so not an orthogonal basis.

Instead, note



Then let
$$\psi_{jk}(x) = \begin{cases} 1, & \frac{2k}{2^j} \leq x < \frac{2k+1}{2^j} \\ -1, & \frac{2k+1}{2^j} \leq x < \frac{2k+2}{2^j} \end{cases}$$

Then let $\psi_{jk}(x) = \begin{cases} -1, & \frac{2k+1}{2^j} \le x < \frac{2k+2}{2^j} \\ 0, & \text{else} \end{cases}$

And replace $\phi_{j+1,2k}$ with $\psi_{j+1,2k}$, so we get an orthonormal basis

$$\left\{ \begin{array}{l} \phi_{00}, \phi_{01}, \phi_{02}, \ldots \\ \psi_{10}, \psi_{12}, \psi_{14}, \ldots \\ \psi_{20}, \psi_{22}, \psi_{24}, \ldots \\ \quad \vdots \end{array} \right\}$$

Furthermore, we can restrict to get an orthonormal basis for functions supported on $[0,1]$

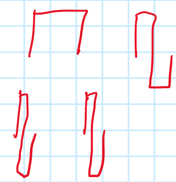
$$\left\{ \begin{array}{l} \phi_{00}, \psi_{10} \\ \psi_{20}, \psi_{22} \\ \psi_{30}, \psi_{32}, \psi_{34}, \psi_{36} \\ \psi_{40}, \psi_{42}, \ldots, \psi_{4,14} \\ \quad \vdots \end{array} \right\}$$

support length 1:

support length $\frac{1}{2}$:

" " $\frac{1}{4}$

" " $\frac{1}{8}$



For any finite support function, can approximate by choosing a scale vector $\phi(x)$ whose scale is that of the support of the function.

Straightforward to approximate it with $\phi_{jk}(x)$ for fixed j.
(getting a 2^j-pt sample)

$$f(x) \approx \sum_{k=0}^{2^j-1} s_k \phi(2^j x - k)$$, which we can write as $(s_0, s_1, \ldots, s_{2^j-1})$

To rewrite in the Haar basis over the unit interval, need to find c_i's

$$
\begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \end{bmatrix}
=
\begin{array}{c}
\begin{bmatrix}
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\
1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\
1 & -1 & 0 & -1 & 0 & 0 & 0 & -1
\end{bmatrix} \\
\begin{matrix} \phi_{00} & \psi_{10} & \psi_{20} & \psi_{22} & \psi_{30} & \psi_{32} & \psi_{34} & \psi_{36} \end{matrix}
\end{array}
\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{bmatrix}
$$

Transform from evenly spaced basis to Haar basis.

Matrix inverses are slow, but here we can do better

$$(s_0 - s_1)/2 = c_4 \qquad (s_2 - s_3)/2 = c_5 \qquad (s_4 - s_5)/2 = c_6 \qquad (s_6 - s_7)/2 = c_7$$

$$\left(\frac{s_0+s_1}{2} - \frac{s_2+s_3}{2}\right)/2 = c_2 \qquad \left(\frac{s_4+s_5}{2} - \frac{s_6+s_7}{2}\right)/2 = c_3$$

$$\left(\frac{s_0+s_1+s_2+s_3}{4} - \frac{s_4+s_5+s_6+s_7}{4}\right)/2 = c_1$$

$$\frac{s_0 + \cdots + s_7}{8} = c_0$$