20. Wavelet systems

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The Haar wavelet was built from a scale function $\phi(x)$ s.t. $\phi(x) = \phi(2x) + \phi(2x-1)$

Consider a scale function solving a more general dilation eq:

$$\phi(x) = \sum_{k=0}^{d-1} c_k \, \phi(2x-k)$$

And let

$$\phi_{jk}(x) = \phi(2^j x - k)$$

Let $V_j = \operatorname{span}\{\phi_{jk}\}_{k\in\mathbb{Z}}$. Then $\phi_{jk} \in V_{j+1}$, so $V_0 \subseteq V_1 \subseteq \cdots$

So we still have the nice property where each successive set of finer-grained ϕ_{jk} spans the coarser resolution span before it.

We will show that it is in general possible to build a wavelet system of orthonormal bases out of a scale function.

Solving a dilation equation

Easy to check a sol $\phi(x) = \sum_{k=0}^{d-1} c_k \, \phi(2x-k)$, harder to find.

Lemma: $\phi(x)$ has support $[0, d-1]$ (if it has finite support)

proof. Say $\phi(x)$ has finite support $[A, B]$.

Then $\phi(2x)$ has support $\left[\frac{A}{2}, \frac{B}{2}\right] \subseteq [A, B]$.

$\Rightarrow A = 0$ (leftmost $\phi(2x-k)$)

Then $\phi(2x-(d-1))$ has support $\left[\frac{d-1}{2}, \frac{d-1}{2} + \frac{B}{2}\right] \subseteq [0, B]$

$\Rightarrow B = d-1$.

$\left[0, \frac{d-1}{2}\right]$

$\left[\frac{1}{2}, \frac{d}{2}\right]$

$\vdots$

$\left[\frac{d-1}{2}, d-1\right]$

Note: $\phi(2x-k)$ has support $\left[\frac{k}{2}, \frac{d-1}{2} + \frac{k}{2}\right]$.

Cascade algorithm:

See better ref [Malone, 2000]

Define an operator $\mathcal{V}$ by

$$(\mathcal{V}f)(x) = \sum_{x=0}^{d-1} c_k \, f(2x-k)$$

Then we are clearly looking for a fixed point of this operator.

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Let f be a compactly supported solution on $[0, N]$, $\quad (N = d-1) \quad \sum_{k=0}^{N} c_k f(2x-k)$

Then $(\mathcal{V}f)(0) = c_0 f(0) + c_1 \cancel{f(-1)} + c_2 \cancel{f(-2)} + \cdots + \cancel{c_N f(-N)} = c_0 f(0)$

$(\mathcal{V}f)(1) = c_0 f(2) + c_1 f(1) + c_2 f(0) \quad = c_2 f(0) + c_1 f(1) + c_0 f(2)$

$(\mathcal{V}f)(2) = c_0 f(4) + c_1 f(3) + c_2 f(2) + c_3 f(1) + c_4 f(0) \quad \vdots$

$\vdots$

$(\mathcal{V}f)(N-1) = c_0 \cancel{f(2N-2)} + \cdots + c_{N-3} f(\overset{N+1}{\cancel{2N-2-N+3}}) + c_{N-2} f(N) + c_{N-1} f(N-1) + c_N f(N-2)$

$(\mathcal{V}f)(N) = c_N f(N)$

Because of support of each $f(2x-k)$

So let $\vec{g} = [f(0), f(1), \ldots, f(N)]^T$

Then $\vec{g} = M\vec{g}$, where matrix M is given by $(\mathcal{V}f)$ above:

$$\begin{bmatrix} c_0 & & & & & & 0 & & \\ c_2 & c_1 & c_0 & & & & & & \\ c_4 & c_3 & c_2 & c_1 & c_0 & & & & \\ & & & & \ddots & & & & \\ & & & & & \ddots & & & \\ & 0 & & & & & c_N & c_{N-2} & c_{N-2} \\ & & & & & & & & c_N \end{bmatrix}$$

M under certain conditions has a nice spectral gap, so this iteration rapidly converges to an approximation $\vec{g}$ of $f(x)$.

Alternate solution by increasing resolution

Consider $\phi(x) = \frac{1}{2}\phi(2x) + \phi(2x-1) + \frac{1}{2}\phi(2x-2)$ with support $[0, 2)$.

$\phi(0) = \frac{1}{2}\phi(0) + \phi(-1) + \frac{1}{2}\phi(-2) = \frac{1}{2}\phi(0) + 0 + 0 \Rightarrow \phi(0) = 0.$

$\phi(2) = \frac{1}{2}\phi(4) + \phi(3) + \frac{1}{2}\phi(2) = \frac{1}{2}\phi(2) \Rightarrow \phi(2) = 0.$

$\phi(1) = \frac{1}{2}\phi(2) + \phi(1) + \frac{1}{2}\phi(0) = \phi(1) \Rightarrow \underbrace{\phi(1) \text{ arbitrary}}_{\text{because can always scale.}}$

Set $\phi(1) = 1.$

Then

$$\phi\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\phi(1) + \phi(0) + \tfrac{1}{2}\phi(-1) = \tfrac{1}{2}$$

$$\phi\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\phi(3) + \phi(2) + \tfrac{1}{2}\phi(1) = \tfrac{1}{2}$$

$$\phi\left(\tfrac{1}{4}\right) = \tfrac{1}{2}\phi\left(\tfrac{1}{2}\right) + \phi\left(-\tfrac{1}{2}\right) + \tfrac{1}{2}\phi\left(-\tfrac{3}{2}\right) = \tfrac{1}{4}$$

$$\vdots$$

Can compute any $\phi\left(\frac{i}{2^j}\right)$ for larger values of j until we ge desired accuracy.

If $\phi(x)$ is simple, can conjecture explicit form and check.

Conditions on dilation equation

(Recall)

Lemma 11.1 If $\phi(x) = \sum_{k=1}^{d-1} c_k (2x-k)$, then either $\sum_{k=0}^{d-1} c_k = 2$ or $\int_{-\infty}^{\infty} \phi(x)\,dx = 0$.

Necessary condition for orthogonality:

Lemma 11.2 Let $\phi(x) = \sum_{k=0}^{d-1} c_k \phi(2x-k)$. If $\phi(x)$ and $\phi(x-k)$ are orthogonal for $k \neq 0$ and $\phi(x)$ has been normalized so

$$\int_{-\infty}^{\infty} \phi(x)\phi(x-k) = \delta(k), \quad \text{then} \quad \sum_{i=0}^{d-1} c_i c_{i-2k} = 2\delta(k).$$

Where $\delta(k)$ is Kronecker delta $\delta(x) = \begin{cases} 1, & x=0 \\ 0, & x \neq 0. \end{cases}$

proof.

$$\delta(k) = \int_{-\infty}^{\infty} \phi(x)\phi(x-k)\,dx$$

$$= \int_{-\infty}^{\infty} \left(\sum_{i=0}^{d-1} c_i \phi(2x-i)\right)\left(\sum_{j=0}^{d-1} c_j \phi(2x-2k-j)\right) dx$$

$$= \sum_{i=0}^{d-1}\sum_{j=0}^{d-1} c_i c_j \int_{-\infty}^{\infty} \phi(2x-i)\,\phi(2x-2k-j)\,dx$$

rescale

$$= \frac{1}{2}\sum_{i=0}^{d-1}\sum_{j=0}^{d-1} c_i c_j \int_{-\infty}^{\infty} \phi(x-i)\,\phi(x-2k-j)\,dx$$

translate

$$= \frac{1}{2}\sum_{i=0}^{d-1}\sum_{j=0}^{d-1} c_i c_j \int_{-\infty}^{\infty} \phi(x)\,\phi(x+i-2k-j)\,dx$$

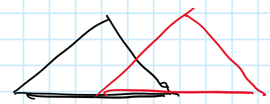
$$= \frac{1}{2}\sum_{i=0}^{d-1}\sum_{j=0}^{d-1} c_i c_j\, \delta(2k+j-i) = \frac{1}{2}\sum_{i=0}^{d-1} c_i c_{i-2k}$$

$$\Rightarrow \sum_{i=0}^{d-1} c_i c_{i-2k} = 2\delta(k)$$

$$\Rightarrow \sum_{i=0}^{d-1} c_i c_{i-2k} = 2\,\delta(k). \qquad \boxed{}$$

For any $k \neq 0$, sum of product of offset coefficients is 0.
If $k=0$, sum of squared coefficients is 2.

Note: The triangular solution to $\phi(x) = \frac{1}{2}\phi(2x) + \phi(2x-1) + \frac{1}{2}\phi(2x-2)$ fails this condition; does not work for constructing orthonormal wavelet basis



Lemma 11.3 If $[0, d-1)$ is the support of $\phi(x)$, and the set of integer shifts, $\{\phi(x-k) \mid k \in \mathbb{Z}\}$ are linearly independent, then $c_k = 0$ unless $0 \le k \le d-1$.

proof. $\phi(x) = \sum_{k=-\infty}^{\infty} c_k\, \phi(2x-k)$. Because $\{\phi(2x-k)\}$ are lin. ind., c_k's are unique

If $\text{support}(\phi(x)) = [0, d-1)$, then $\text{support}(\phi(2x)) = [0, \frac{d-1}{2})$.

So $\text{support}(\phi(2x-k)) = [\frac{k}{2}, \frac{k}{2} + \frac{d-1}{2})$

But $\text{support}(\phi(x)) = \text{support}\left(\sum_{-\infty}^{\infty} c_k\, \phi(2x-k)\right) = \bigcup_{k=-\infty}^{\infty} c_k \,\text{support}(\phi(2x-k))$

But since $\phi(2x-k)$ are lin. ind., $c_k\phi(2x-k)$ for $k \ge d$ or $k<0$ cannot be "cancelled out" by other terms in the summation if $c_k > 0$.

So, $\phi(x) = \sum_{k=0}^{d-1} c_k\, \phi(2x-k)$

Corollary: d is even, (assuming linearly ind. + orthogonal)

proof. Suppose d is odd.

By Lemma 11.2, $\sum_{i=0}^{d-1} c_i c_{i-2k} = 0$ for $k \neq 0$.

Let $k = \frac{d-1}{2}$. Then $0 = \sum_{i=0}^{d-1} c_i c_{i-d+1} = c_{d-1} c_0 \Rightarrow c_0$ or c_{d-1} is 0

Then only $d-1$ nonzero coefficients, so we just shift so that d is even $\boxed{}$

Thus, if a dilation equation has d terms, where d is even, the coefficients have to satisfy.

$$\sum_{i=0}^{d-1} c_i = 2, \quad \underbrace{\sum_{i=0}^{d-1} c_i c_{i-2k} = 2\delta(k)}$$

to satisfy.

$$\underbrace{\sum_{i=0} c_i = 2}_{1 \text{ linear eq}}, \quad \underbrace{\sum_{i=0} c_i c_{i-2k} = 2\delta(k)}_{\frac{d}{2} \text{ quadratic equations for } 0 \le k \le \frac{d}{2}-1}$$

But, we have d degrees of freedom, so roughly speaking, we have $\frac{d}{2}-1$ leftover degrees of freedom to design the wavelet for our desired properties.