26. Complexes

Tuesday, November 16, 2021 7:45 PM

In order to formally approach persistent homology, we need some more machinery.
Based off notes from Melissa McGuirl, as well as Edelsbrunner & Harer

Def 3.1 Consider the set of points $\{u_i\}_{i=0}^n$.

An **affine combination** is a point $x = \sum_{i=0}^n \lambda_i u_i$ s.t. $\sum_{i=0}^n \lambda_i = 1$.

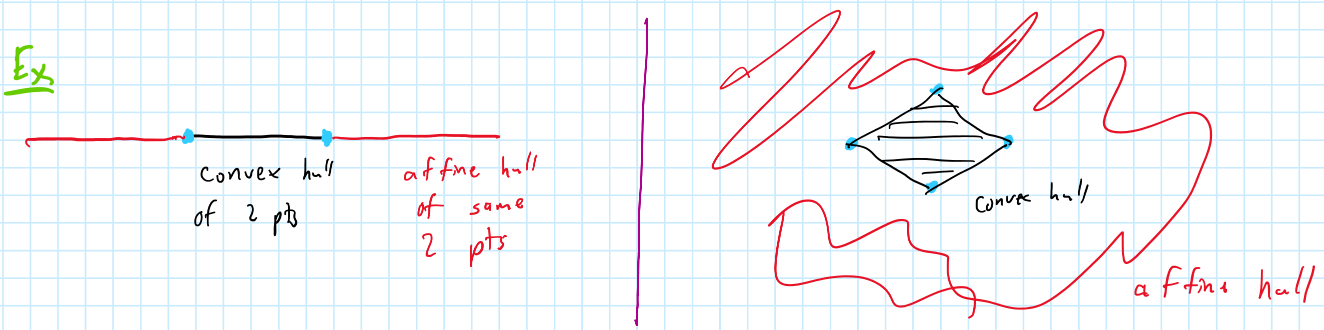
A **convex combination** is a point $x = \sum_{i=0}^n \lambda_i u_i$ s.t. $\sum_{i=0}^n \lambda_i = 1$ and $\lambda_i \geq 0 \ \forall i$.

Def. 3.2 Affine and convex hulls.

$$\operatorname{aff}(u_0, \dots, u_n) = \left\{ x = \sum_{i=0}^n \lambda_i u_i \ \middle|\ \sum_{i=0}^n \lambda_i = 1 \right\}$$

$$\operatorname{conv}(u_0, \dots, u_n) = \left\{ x = \sum_{i=0}^n \lambda_i u_i \ \middle|\ \sum_{i=0}^n \lambda_i = 1,\ \lambda_i \geq 0 \ \forall i \right\}$$

Ex

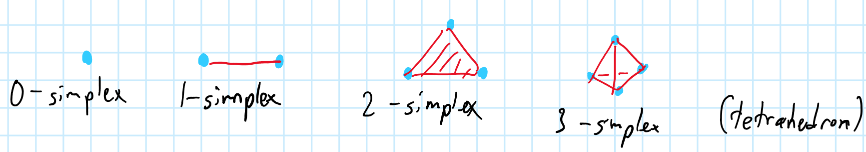


Def. 3.3 $u_0, \dots, u_n$ are **affinely ind.** iff the n vectors $u_i - u_0$ for $1 \leq i \leq n$ are linearly ind.

Ex. In $\mathbb{R}^d$, at most $d+1$ affinely ind. pts.

Def. 3.4 A **k-simplex** is the convex hull of $k+1$ affinely ind. points
$\sigma = \operatorname{conv}(u_0, \dots, u_n)$, $\dim(\sigma) = n$.

Ex.



Def. 3.5 Given $\sigma = \operatorname{conv}(u_0, \dots, u_n)$, a **face** τ of σ, denoted $\tau \leq \sigma$
is $\tau = \operatorname{conv}(u_{i_1}, \dots, u_{i_m})$, where $\{u_{i_1}, \dots, u_{i_m}\} \subset \{u_0, \dots, u_n\}$.

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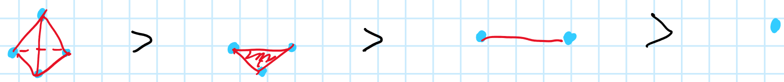
τ is a **proper face** if $m < n$.

Ex.



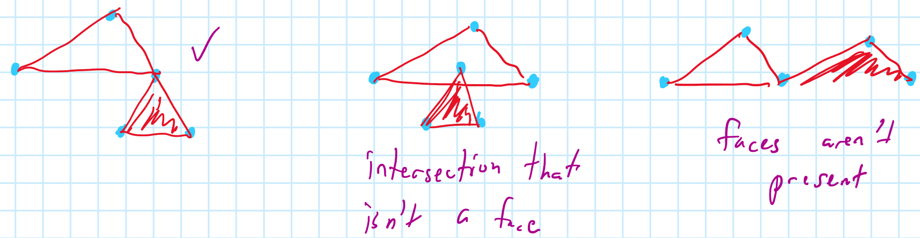
Def. 3.6 A **simplicial complex** is a finite collection of simplices K s.t.

(1) $\sigma \in K$ and $\tau \leq \sigma \Rightarrow \tau \in K$

(2) $\sigma_1, \sigma_2 \in K \Rightarrow$ either (i) $\sigma_1 \cap \sigma_2 = \emptyset$ or

(ii) $\sigma_1 \cap \sigma_2 \leq \sigma_1$ and $\sigma_1 \cap \sigma_2 \leq \sigma_2$.

Ex.

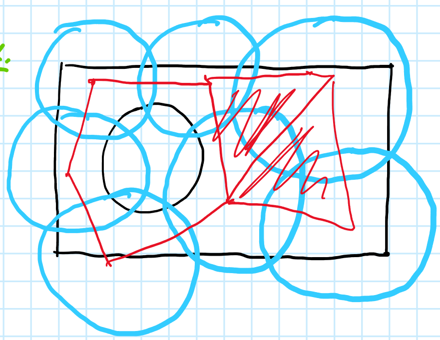


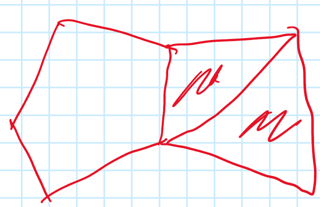
Def. 3.7 An **abstract simplicial complex** is a finite collection of sets A s.t. $\alpha \in A$ and $\beta \subset \alpha$ implies $\beta \in A$.

Def. 3.8 Let X be a topological space. A **cover** of X is a collection of sets $\mathcal{U} = \{U_i\}_{i \in I}$ s.t. $X \subset \bigcup_{i \in I} U_i$.

Def. 3.9 Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a cover of X. The **nerve** of $\mathcal{U}$, $\mathcal{N}(\mathcal{U})$, is the abstract simplicial complex with vertex set I, where a family $\{i_0, \dots, i_k\}$ spans a k-simplex iff $U_{i_0} \cap \dots \cap U_{i_k} \neq \emptyset$.

Ex.





Def. Given continuous maps $f, g : X \to Y$, a homotopy between f and g is another continuous map $H : X \times [0,1] \to Y$ s.t. $f(x) = H(x, 0)$
$g(x) = H(x, 1)$ $\forall x \in X$.

If such a map H exists, then $f \simeq g$, and call f and g homotopic.

Two topological spaces X and Y are homotopy equivalent if there are continuous maps $f : X \to Y$ and $g : Y \to X$ s.t. $g \circ f \simeq id_X$, $f \circ g \simeq id_Y$.

Note that homotopy is a stronger notion than homology, which we'll discuss later. Unfortunately, homotopy is hard to compute.
Note also that homotopy-equivalence is weaker than topological equivalence.

Thm 3.1 (Nerve Thm) Let U be a finite collection of closed, convex sets in Euclidean space. Then $\mathcal{N}(U)$ and the union of the sets in U have the same homotopy type.

Of course, we still need to construct appropriate covers to get a simplicial complex

Def. 3.10 (Cech complex)
Let X be a finite set of points in $\mathbb{R}^d$. For each $x \in X$,
let $B_r(x) = \{ y \in \mathbb{R}^d \mid d(x,y) \leq r \}$ be the closed ball centered at x with radius $r \geq 0$.
The Cech complex of X and r is the nerve of $\{B_r(x)\}_{x \in X}$. i.e

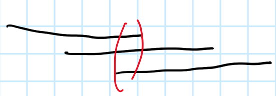
$$Cech(X, r) = \{ \sigma \subset X \mid \bigcap_{x \in \sigma} B_r(x) \neq \emptyset \}$$

Because closed balls are closed in $\mathbb{R}^d$, the Nerve Thm applies.
Note that the vertex set of $Cech(X, r)$ is just all of X

Computing the Cech complex:

Helly's Thm: Let F be a finite collection of closed, convex sets in $\mathbb{R}^d$. Every $d+1$ of the sets have a non-empty intersection iff they all have a non-empty intersection.

Ex.



proof. Induction over d and number of sets $n = |F|$.

- **Base case:** $d = 1$, $\forall n$

Convex sets on the real line are closed intervals $I_1, \ldots, I_n$.

Forward case: Every pair of sets intersect.

let $I_i = [a_i, b_i]$. Then $\bigcap_i I_i = [\max_i a_i, \min_i b_i]$

If $\max_i a_i > \min_i b_i$, then $\exists\, a_i > b_j$ for some i & j.

But then $I_i \cap I_j = \emptyset$, a contradiction.

Backward case: $\bigcap_i I_i \subseteq I_i \cap I_j \quad \forall i, j$ clearly.

(backward case is obvious for all cases actually)

$d \backslash n$	1	2	3	4	5
1	✓	✓	✓	✓	✓
2	✗	✗	✓		
3	✗	✗	✗	✓	
4	✗	✗	✗	✗	✓
5	✗	✗	✗	✗	✗

- **Base case:** $n = d+1$. Trivial by definition.

Irrelevant case: $n \le d$. Not meaningful since impossible to have $d+1$ sets.

General case: Suppose $\exists\, n > d+1$ closed convex sets in $\mathbb{R}^d$, denoted $X_1, \ldots, X_n$, that form a minimal counterexample, where every $d+1$ of the sets has a common intersection, but not all n sets.

Inductive hypothesis: True for $(d, n-1)$ and $(d-1, n)$.

By minimality & the inductive hypothesis,

$$Y_n = \bigcap_{i=1}^{n-1} X_i \text{ is non-empty \& disjoint from } X_n.$$

Because Y_n and X_n are closed and convex, $\exists\, (d-1)$-dim hyperplane h separating them & disjoint from both sets.

Let F' be the collection of sets $Z_i = X_i \cap h$, for $1 \le i \le n-1$.

Note: each Z_i is a non-empty $(d-1)$-dim closed, convex set because by assumption, d of the first n-1 sets X_i have a common intersection with X_n.

Thus, that common intersection of d sets contains points on both sides of h, since they intersect both Y_i and X_i.

$\Rightarrow$ any d sets of $\{Z_i\}$ have a common intersection,

$$\Rightarrow \bigcap F' \neq \emptyset \quad \text{(by inductive hypothesis)}$$

But, $\bigcap F' = \bigcap_{i=1}^{n-1} (X_i \cap h) = Y_n \cap h$, a contradiction. $\boxtimes$

But, $\cap F' = \bigcap_{i=1}^{n-1} (X_i \cap h) = Y_n \cap h$, a contradiction.

Let's go back to computing the Cech complex.

Note that a set of balls of equal radius has a non-empty intersection iff their centers lie in a ball of the same radius.

$\Rightarrow$ y belongs to all balls iff $d(x,y) \le r$ for all centers $x \in X$.

Corollary: (Jung's thm) Let $X \subseteq \mathbb{R}^d$ finite. Every $d+1$ point in X are contained in a common ball of radius r iff all points in X are

Let $\sigma \subseteq X$. Then $\sigma \in Cech(X, r)$ if $\sigma \subseteq B_r(y)$ for some $y \in \mathbb{R}^d$.

Let miniball(σ) be the smallest closed ball containing σ (which is unique)

The radius of miniball$(\sigma) < r \iff \sigma \in Cech(X, r)$

Next time: we show how to compute miniball(σ)