# 27. Complexes and chains

Wednesday, November 17, 2021 12:59 AM

**Last time:** Defined Čech complexes

**Today:**
- Compute Čech complexes
- Define Vietoris Rips complex
- Introduce chain complexes & homology.

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**Recall:** We need to compute minball($\sigma$) to determine if $\sigma \in \text{Čech}(X, r)$.

**Note:** A miniball is determined by boundary pts, so can remove interior points.

**Algorithm** returns miniball with $\tau$ in the interior and $\nu$ on the boundary.

```
def MiniBall(τ, ν):
    if τ = ∅, then compute the miniball B of ν directly.     (e.g. can find center
                                                               by minimizing squared
    else, choose a random u ∈ τ                                deviation)
        B = miniball(τ - {u}, ν)        (remove u from interior)
        if u ∉ B, then B = miniball(τ - {u}, ν ∪ {u})   (put u on boundary if necessary)
    return B
```

Then $\text{miniball}(\sigma, \emptyset) = \text{miniball}(\sigma)$.

Each iteration reduces $\tau$ by 1, at the cost of possibly 2 recursive calls.

Possibly $2^n$ time unless we can control "if $u \notin B$".

Let $t_j(n)$ be the expected computational complexity with $n$ pts in $\tau$ and $j = d+1 - |\nu|$ possibly open positions on the boundary.

$t_j(0) = 0$ (obviously) (0 calls to miniball)

If $n > 0$, then $\text{Prob}(u \notin B) = \text{Prob}(u \text{ needs to be a boundary element}) \leq \frac{j}{n}$.

Thus, $t_j(n) \leq \underbrace{t_j(n-1)}_{\text{miniball}(\tau - \{u\}, \nu)} + \underbrace{1}_{u \notin B} + \frac{j}{n} \underbrace{t_{j-1}(n-1)}_{\text{miniball}(\tau - \{u\}, \nu \cup \{u\})}$

$t_0(n) \leq t_0(n-1) + 1 \implies t_0(n) \leq n$

$t_1(n) \leq t_1(n-1) + 1 + \frac{1}{n} t_0(n-1) \leq t_1(n-1) + 2 \implies t_1(n) \leq 2n$

$t_2(n) \leq t_2(n-1) + 1 + \frac{2}{n} \overbrace{t_1(n-1)}^{2n} \leq t_2(n-1) + 5 \implies t_2(n) \leq 5n \leq 6n \leq 3! \cdot n$

$$t_3(n) \le t_3(n-1) + 1 + 3 \cdot 3! \quad \Rightarrow \quad t_3(n) \le 4! \cdot n$$

$$t_4(n) \le (1 + 4 \cdot 4!)\, n = 5! \cdot n$$

$$\vdots$$

$$t_j(n) \le (j+1)! \cdot n$$

But $j \le d+1$ because at most $d+1$ boundary pts, so for constant dim, alg takes $O(n)$ time to compute a miniball.

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The Cech complex checks all subcollections, which is slow.
We can approximate by just checking pairs.

**Def. 3.4** Let $X \subseteq \mathbb{R}^d$ finite set of pts.

The **Vietoris-Rips complex** of $X$ and $r$ is defined by

$$VR(X, r) = \{ \sigma \subset X \mid B_r(x_i) \cap B_r(x_j) \neq \emptyset \quad \forall x_i, x_j \in \sigma \}.$$

i.e. $VR(X,r)$ contains all subsets of $X$ with diameter no greater than $2r$.

Also, it is easy to see $Cech(X, r) \subset VR(X, r)$.

Exercise: Prove $VR(X, r) \subseteq Cech(X, r\sqrt{2})$

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There are a number of other ways to build a simplicial complex on a finite metric space, including Delaunay complexes, alpha complexes, Witness complexes, etc.
But for now, let's turn to homology.

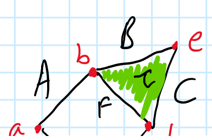
**Def.** Let $K$ be a simplicial complex. An **$i$-chain** is a formal sum of $i$-simplices $\sum c_i \sigma_i$, where $c_i \in \mathbb{F}$ and the sum is taken over all possible $i$-simplices $\sigma_i \in K$. The set of all $i$-chains is denoted $C_i(K)$

Often, we let $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

$C_i(K)$ is a vector space over $\mathbb{F}$, called the vector space of $i$-chains in $K$.

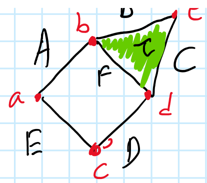
Note, the $i$-simplices form a basis of $C_i(K)$, so $\dim(C_i(K)) = \#\, i$-simplices.

**Ex.**




0-simplices $\{a, b, c, d, e\}$
1-simplices $\{A, B, C, D, E, F\}$

Ex.




0-simplices $\{a, b, c, d, e\}$

1-simplices $\{A, B, C, D, E, F\}$

2-simplices $\{\tau\}$

$C_0(K) = \langle a, b, c, d, e \rangle$ $\quad \ni a+c+e$

$C_1(K) = \langle A, B, C, D, E, F \rangle$ $\quad \ni A+B+D+E$

$C_2(K) = \langle \tau \rangle$ $\quad \ni \tau$

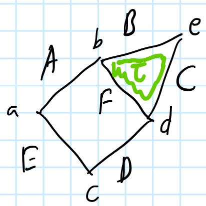
If $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

**Definition (boundary of simplex)** Let $\sigma = [u_0, u_1, \dots, u_k]$ be a $k$-simplex. The boundary of $\sigma$ is a map $\partial_k : C_k(K) \to C_{k-1}(K)$

$$\partial_k \sigma = \sum_{i=0}^{k} [u_0, u_1, \dots, \hat{u}_i, \dots, u_k],$$

where we use the notation $\hat{u}_i$ to indicate that $u_i$ is omitted.

Ex.




$\partial(\tau) = B + C + F$

$\partial(A) = a + b$

**Def. 4.3 (Chain complex)**

A chain complex is a sequence of chain groups connected by boundary maps

$$\cdots \xrightarrow{\partial_{i+2}} C_{i+1}(K) \xrightarrow{\partial_{i+1}} C_i(K) \xrightarrow{\partial_i} C_{i-1}(K) \xrightarrow{\partial_{i-1}} \cdots$$

Ex.

$$\emptyset \xrightarrow{\partial_3} \langle \tau \rangle \xrightarrow{\partial_2} \langle A, B, C, D, E, F \rangle \xrightarrow{\partial_1} \langle a, b, c, d, e \rangle \xrightarrow{\partial_0} \emptyset.$$

**Def. 4.4 (i-cycle)**

An $i$-chain $c$ is an $i$-cycle if $\partial_i c = 0$.

Ex. $\partial(C+B+F) = e+d + d+b + b+e = 2d + 2e + 2b = 0$

$\Rightarrow C+B+F$ is an $i$-chain.

**Def. 4.5** (i-boundary)

An i-chain $c$ is an i-boundary if there exists an $i+1$-chain $d \in C_{i+1}(K)$ s.t. $c = \partial_{i+1}(d)$.

**Ex.** $B + C + F = \partial(\tau)$.

**Lemma 4.1** (Fundamental Lemma of homology)

$$\partial_p \circ \partial_{p+1}(d) = 0 \quad \forall p \in \mathbb{Z} \text{ and for all } i\text{-chains } d.$$

**proof.** We need only show this for $(p+1)$-simplex $\tau$. i.e. $\partial_p \circ \partial_{p+1}(\tau) = 0$.

The boundary $\partial_{p+1}\tau$ consists of all $p$-faces of $\tau$.

Every $(p-1)$-face of $\tau$ belongs to exactly two $p$-faces,

so $\partial_p(\partial_{p+1}\tau) = 0$. $\boxed{/}$