28. Simplicial homology

Thursday, November 18, 2021 1:49 AM

Last time: Defined chain complexes

Today: Computing simplicial homologies

Let K be a simplicial complex

Let $C_p = C_p(K)$ be the group of p-chains,

where $c = \sum a_i \sigma_i$, where $\sigma_i \in K$ are p-simplicies, for $c \in C_p$ and $a_i \in \mathbb{Z}_2$.

$\partial_p : C_p \to C_{p-1}$ is given by $\partial_p \sigma = \sum_{j=0}^{p} [u_0, \ldots, \hat{u}_j, \ldots, u_p]$ the boundary homomorphism.

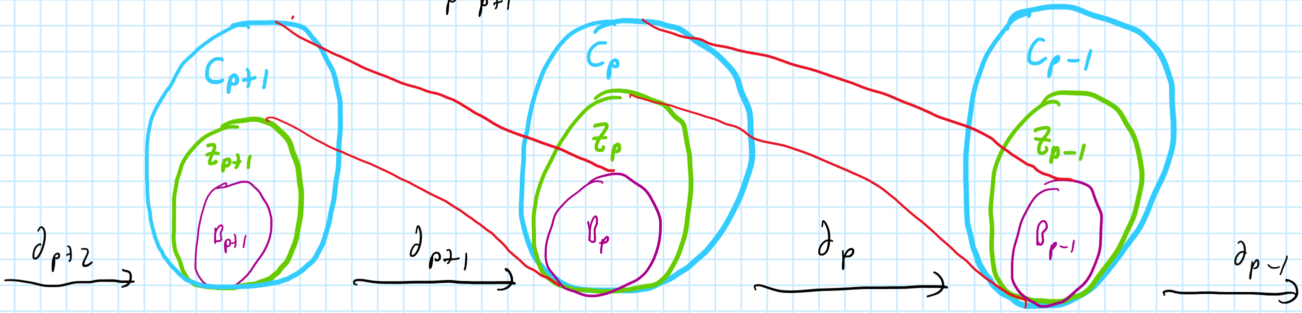
Giving rise to the chain complex

$$\cdots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} C_{p-2} \longrightarrow \cdots$$

Let $Z_p = Z_p(K) = \{ c \in C_p \mid \partial_p c = 0 \} = \operatorname{Ker} \partial_p$, the subgroup of p-cycles.

Let $B_p = B_p(K) = \{ \partial_{p+1} d \mid d \in C_{p+1} \} = \operatorname{Im} \partial_{p+1}$, the subgroup of p-boundaries.

Fundamental Lemma: $\partial_p \partial_{p+1} d = 0$.



Everything in Z_{p+1} gets mapped to 0 and everything in C_{p+1} goes to B_p.

The boundary group B_p is a subgroup of the cycle group Z_p by the fundamental lemma.

Definition The **pth homology group** is the pth cycle group modulo the pth boundary group

$$H_p = Z_p / B_p$$

The **pth Betti number** is the rank of this group, $\beta_p = \underbrace{\text{rank}}_{\text{smallest cardinality of generating set}} H_p$

Recall: For $c \in Z_p$, the cosets $c + B_p$ form H_p.

Definition: The cosets of H_p are referred to as a **homology class**, and any $c_1, c_2 \in c + B_p$ are **homologous**, denoted $c_1 \sim c_2$.

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Recall: The cardinality of a group is called its *order*

So $C_p = \langle \sigma_1, \sigma_2, \dots, \sigma_n \rangle$, where $\sigma_i \in K$ are p-simplices.

$$\Rightarrow \operatorname{ord}(C_p) = |C_p| = 2^n$$

Note $C_p \cong \mathbb{Z}_2^n$, the group of length-n bit vectors under XOR.

Recall: rank of a vector space is its dimension, so $\operatorname{rank}(C_p) = n$.

Then $\beta_p = \operatorname{rank} H_p = \log_2 |H_p| = \log_2 \dfrac{|Z_p|}{|B_p|} = \operatorname{rank} Z_p - \operatorname{rank} B_p$

Ex. Let K be a triangulation of $\mathbb{B}^k = \{x \in \mathbb{R}^k \mid |x| \leq 1\}$.

Then $H_p(K) = \{0\} \quad \forall p \neq 0$, and $\beta_0 = 1$.

(hard to prove, but makes sense as no "holes")

Simpler Let K be the faces of a single k-simplex σ_k.

Claim: $H_p(K) = \{0\} \quad \forall p \neq 0$ and $\beta_0 = 1$.

proof. $H_p(K) = \{0\} \iff Z_p = B_p$

i.e. we need to show that all p-cycles are p-boundaries for $p > 0$.

Let $\{u_0, \dots, u_k\}$ be the set of vertices.

- Note $C_k = \{0, \sigma_k\}$ and $C_{k+1} = \{0\}$. $\Rightarrow B_k = \{0\}$. (image of 0 under ∂)

  $$\partial \sigma_k = \sum_{j=0}^{k} [u_0, \dots, \hat{u}_j, \dots, u_k] \neq 0$$ because each k-simplex appears exactly once.

  $$\Rightarrow \sigma_k \notin Z_k \Rightarrow Z_k = \{0\} \Rightarrow H_k = \{0\}.$$

- Let's consider $0 < p < k$

  Let $c \in Z_p$ be a p-cycle with simplices of the form $[u_{i_0}, \dots, u_{i_p}]$

  Let d be the set of all $p+1$-simplices of the form $[u_0, u_{i_0}, \dots, u_{i_p}]$

  Note that if u_0 is already in a simplex of c, there is no corresponding $p+1$-simplex.

  We can also view $d \in C_{p+1}$ as a $p+1$-chain. (rather than just a set)

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we make use of K being the faces of σ_k as otherwise might not be in C_{p+1}.

Claim: $\partial d = c$.

proof.

Case 1a: Consider p-simplex $\tau \in c$ where $u_0 \notin \tau$.

Then u_0 occurs exactly once as a face of $[u_0, u_{i_0}, \ldots, u_{i_p}] \in d$,

so τ appears exactly once in ∂d.

Case 1b: Consider p-simplex $\tau \in c$ where $u_0 \in \tau$.

Let σ be the $p-1$-simplex formed by dropping u_0 from τ

We know σ appears an even number of times in ∂c, as $c \in Z_p$

$\Rightarrow$ an even number of p-simplices in c contain σ.

If a p-simplex contains both u_0 and σ, it must be τ.

All other p-simplices containing σ have a corresponding $p+1$-simplex in d,

$\Rightarrow$ odd # of $p+1$-simplices in d that give rise to τ under ∂.

$\Rightarrow$ τ appears ~~an~~ odd number of times in ∂d.

Case 2a: Consider p-simplex $\tau \notin c$ where $u_0 \in \tau$.

Let σ be the $p-1$ simplex formed by dropping u_0 from τ, as above in Case 1b.

$\Rightarrow$ even number of p-simplices in c contain σ.

But none of these p-simplices can contain both u_0 and σ, as $\tau \notin c$.

$\Rightarrow$ All of them have corresponding $(p+1)$-simplex in d.

$\Rightarrow$ τ appears an even number of times in ∂d (and $= 0$)

Case 2b: Consider p-simplex $\tau \notin c$ where $u_0 \notin \tau$.

In order for $\tau \in \partial d$, must exist some vertex u' s.t. $[u', \tau] \in d$.

But $u' \neq u_0$, because otherwise $[u_0, \tau] \in d \Rightarrow \tau \in c$ by construction of d, so contradiction.

And if $u' \neq u_0$, then $u_0 \notin [u', \tau] \in d$, which also contradicts the construction of d.

$\Rightarrow$ $\tau \notin \partial d$.

$\boxtimes$

$\Rightarrow$ $\partial d = c$.

Hence $Z_p \subseteq B_p$ $\Rightarrow$ $Z_p = C_p$ for $0 < p < k$.

- Consider now $p=0$.

  Note that the boundary of any vertex is 0.
  So $Z_0 = C_0$, $|Z_0| = 2^{k+1}$

  Suppose we have a 0-cycle $c = u_{i_1} + \cdots + u_{i_\ell}$.

  If ℓ is even, we can pair off vertices to form $d \in C_1$ s.t. $\partial d = c \Rightarrow c \in B_0$.

  If ℓ is odd, we cannot, so $c \notin B_0$.

  Thus, $|B_0| = \frac{|C_0|}{2} \Rightarrow |H_0| = \frac{|Z_0|}{|B_0|} = 2 \Rightarrow H_0 \cong \mathbb{Z}_2 \Rightarrow \beta_0 = 1$.

It is possible to define a reduced homology $\tilde{\beta}_p$ so that $\tilde{\beta}_p = \beta_p$ for $p > 0$ and $\tilde{\beta}_0 = 0$ for a simplex, which is more convenient sometimes since we may want $\tilde{\beta}_0$ to correspond to some kind of hole instead of just # of connected components

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Def. The Euler characteristic χ of a simplicial complex K is the alternating sum of the number of p-simplices in K.

$$\chi = n_0 - n_1 + n_2 - n_3 + \cdots, \text{ where } n_i \text{ is the \# of } i\text{-simplices, and } n_p = \operatorname{rank} C_p$$

Recall: $\chi = V - E + F$ (vertices $-$ edges $+$ faces)

Notation: Let $z_p = \operatorname{rank} Z_p$ (rank of cycle group)

$b_p = \operatorname{rank} B_p$ (rank of boundary group)

Then $n_p = z_p + b_{p-1}$ (because ∂_p has kernel Z_p and maps onto B_{p-1})

So $\chi = \sum_{p \geq 0} (-1)^p (z_p + b_{p-1})$ (say $b_{-1} = 0$ for notational simplicity, as $C_0 = Z_0$)

$$= z_0 + \overset{\overset{0}{\shortparallel}}{b_{-1}} - z_1 - b_0 + z_2 + b_1 - z_3 - b_2 + \cdots = \sum_{p \geq 0} (-1)^p (z_p - b_p) = \sum (-1)^p \beta_p$$

Fact: Simplicial homology is independent of triangulation of a topological space, and is equivalent to "singular homology." (Beyond scope of this course)

Euler-Poincare Thm The Euler characteristic of a topological space is the alternating sum of its Betti numbers $\chi = \sum_{p \geq 0} (-1)^p \beta_p$.

How do we compute the simplicial homology in general?

$p=0$

How do we compute the simplicial homology in general?

Boundary matrices

Let K be a simplicial complex

Index the p-simplices $x_1, \dots, x_{n_p}$

Index the $p-1$-simplices $y_1, \dots, y_{n_{p-1}}$

$$\partial(x_j) = \sum_{i=1}^{n_{p-1}} a_j^i y_i, \quad \text{where } a_j^i = 1 \text{ if } y_i \text{ is a face of } x_j$$
$$a_j^i = 0 \text{ otherwise.}$$

Then for any p-chain $c = \sum_{j=1}^{n_p} a_j x_j$,

$$\partial_p c = \begin{bmatrix} a_1^1 & a_1^2 & \cdots & a_1^{n_p} \\ a_2^1 & a_2^2 & \cdots & a_2^{n_p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n_{p-1}}^1 & a_{n_{p-1}}^2 & \cdots & a_{n_{p-1}}^{n_p} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n_p} \end{bmatrix},$$ where written in coordinates of y_i.

We will use a generalization of row-echelon form on the boundary matrix.

Note that the cols of ∂_p span $\operatorname{Im}(\partial_p) = B_{p-1}$, so $\operatorname{rank}(\partial_p) = b_{p-1}$

Recall: Exchanging cols or adding one col to another does not change rank
" rows " " " row " " " " " " " } Gaussian elimination

Define: For any matrix $A \in R^{m \times n}$, where R is a principal ideal domain, (incl. fields) there exist $S \in R^{m \times m}$ and $T \in R^{n \times n}$ invertible matrices such that

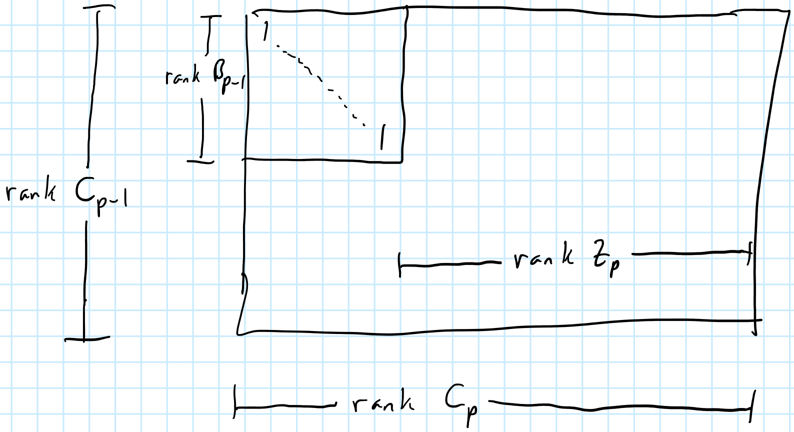
$$SAT = \begin{bmatrix} \alpha_1 & & & & \\ & \alpha_2 & & & \\ & & \ddots & & \\ & & & \alpha_r & \\ & & & & \\ \end{bmatrix}, \quad \text{and } \alpha_i \mid \alpha_{i+1} \text{ for } 1 \le i < r.$$

This is known as the **Smith Normal Form** of A.

Note: S can be thought of as a change of basis in R^m
T " " " " " " " " " R^n

$T \quad T$

Recall $n = z + b$



Recall $n_p = z_p + b_{p-1}$

It turns out we can read the ranks of the boundary and cycle groups off this matrix

Gaussian elimination takes $O(n_{p-1} n_p \min\{n_{p-1}, n_p\})$ time — cubic time
$O((n_{p-1} + n_p)^2)$ space — quadratic memory

} in simplices

This is why simplicial homologies are relatively easy to compute.

For any data sets, we can generate a filtration of Vietoris-Rips complexes by increasing radii, and then image which homologies are persisted.