29. Morse functions

Tuesday, November 23, 2021 12:16 PM

Previously: We looked at simplicial complexes and homology.
We could use Cech and Vietoris-Rips complexes to understand the topology of a point-cloud dataset.

Today: Introduction to Morse theory.
Allows us to understand the homology of a manifold/spac given by a function.

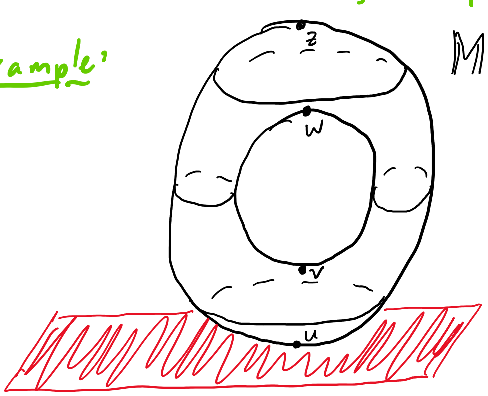
Caveat: I am learning all of this alongside you, as it is not something I have ever personally used.

Definition: A **complex** is a decomposition of a topological space into simple pieces where the common intersections are lower-dim pieces of the same type. e.g. simplicial complexes.

Today, we will give the preliminaries to build up to the Morse-Smale complex.

[Edelsbrunner + Harer, Computational Topology, Ch 6]

Example:



$\mathbb{M}$ = Upright 2D (hollow) torus sitting in $\mathbb{R}^3$.

Let $f(x)$ be the height of a point $x \in \mathbb{M}$.
$f: \mathbb{M} \to \mathbb{R}$ is a height function (simple function on manifold)

$f^{-1}(a)$ is a level set.

Sublevel set $\mathbb{M}_a = f^{-1}(-\infty, a] = \{x \in \mathbb{M} \mid f(x) \leq a\}$.

What happens to the homotopy type of the sublevel set as a increases?

For $a < f(u)$, $\mathbb{M}_a = \emptyset$.

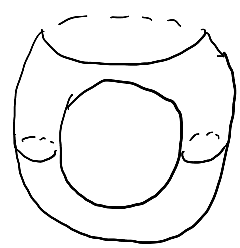
For $f(u) < a < f(v)$, $\mathbb{M}_a$ is a disk.

For $f(v) < a < f(w)$, $\mathbb{M}_a$ is a cylinder.

glue a 1-handle to disk

For $f(w) < a < f(z)$, $\mathbb{M}_a$ is a capped torus

glue a 1-handle to cylinder



For $f(z) < a$, $\mathbb{M}_a$ is entire torus. Glue a disc to capped torus

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[Zomorodian, Topology for computing, Ch 5] simpler intuitive exposition

Let $\mathbb{M}$ be a smooth, compact, 2D manifold without boundary, a **surface**.

Assume for simplicity $\mathbb{M} \subseteq \mathbb{R}^3$, inheriting the subspace topology and Euclidean metric.

Def. 5.1 A **tangent vector** v_p to $\mathbb{R}^3$ consists of two points of $\mathbb{R}^3$:

(1) vector part v

(2) point of application p

The **tangent space** $T_p(\mathbb{R}^3)$ is all tangent vectors to $\mathbb{R}^3$ at p.

Note: $T_p(\mathbb{R}^3)$ is isomorphic to $\mathbb{R}^3$, but there is a different tangent space at each point.

We may also attach tangent spaces to each point in a manifold.

Def. 5.2 Let $p \in \mathbb{M} \subseteq \mathbb{R}^3$. A **tangent vector** v_p to $\mathbb{R}^3$ is **tangent** to $\mathbb{M}$ at p if v is the velocity of some curve in $\mathbb{M}$ at p.

The **tangent plane** $T_p(\mathbb{M})$ is the set of all such tangent vectors.

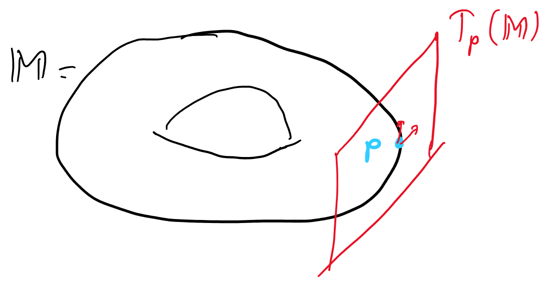
Recall: You can cover a 2-manifold with a number of **charts**, which homeomorphically map a neighborhood of a pt to an open subset of $\mathbb{R}^2$.

The inverses of the maps are charts and can be used to parametrize neighborhoods.

Thm 5.1 Let $p \in \mathbb{M} \subseteq \mathbb{R}^3$, and let φ be a patch in $\mathbb{M}$ s.t. $\varphi(u_0, v_0) = p$. A tangent vector v to $\mathbb{R}^3$ at p is tangent to $\mathbb{M}$ iff

$$v = c_1 \underbrace{\varphi_u(u_0, v_0)}_{\text{directional derivative}} + c_2 \underbrace{\varphi_v(u_0, v_0)}.$$

Thus, the tangent plane $T_p(\mathbb{M})$ is a subspace of the tangent space $T_p(\mathbb{R}^3)$, Thus, the tangent plane $T_p(\mathbb{M})$ is a subspace of the tangent space $T_p(\mathbb{R}^3)$, and is the best linear approximation of the surface $\mathbb{M}$ near p.



Def. 5.3 A **vector field** or **flow** on $\mathbb{M}$ is a function that assigns a vector $v_p \in T_p(\mathbb{M})$ to each pt $p \in \mathbb{M}$.

Def. 5.4 Let $v_p \in T_p(\mathbb{M})$ and let $h: \mathbb{M} \to \mathbb{R}$. The **derivative** $v_p[h]$ of h w.r.t. v_p is the common value of $\frac{d}{dt}(h \circ \gamma)(0)$ for all curves $\gamma \in \mathbb{M}$ with initial velocity v_p. ← note, using Euclidean metric

i.e. we have defined a rate of change for h when going in any tangent direction along the manifold.

Def. 5.5 The **differential** dh_p of $h: \mathbb{M} \to \mathbb{R}$ at $p \in \mathbb{M}$ is a linear function $dh_p : T_p(\mathbb{M}) \longrightarrow \mathbb{R}$ s.t. $dh_p(v_p) = v_p[h]$.

The differential is a machine that converts vector fields into real-valued functions.

We are interested in the geometry that h gives to our manifold.

Def. 5.6 A point $p \in \mathbb{M}$ is **critical** for map $h: \mathbb{M} \to \mathbb{R}$ if dh_p is the zero map. Otherwise, p is **regular**.

i.e. if all the partial derivatives are 0.

Note: We have generalized ordinary multivariable calculus to calculus on manifolds

As with ordinary calculus, we can classify critical points.

Def. 5.7 Let x, y be a patch on $\mathbb{M}$ at p. The Hessian of $h: \mathbb{M} \to \mathbb{R}$ is

$$H(p) = \begin{bmatrix} \frac{\partial^2 h}{\partial x^2}(p) & \frac{\partial^2 h}{\partial y \partial x}(p) \\ \frac{\partial^2 h}{\partial x \partial y}(p) & \frac{\partial^2 h}{\partial y^2}(p) \end{bmatrix}$$

$$H(p) = \begin{bmatrix} \cdots & \cdots \\ \frac{\partial^2 h}{\partial x \partial y}(p) & \frac{\partial^2 h}{\partial y^2}(p) \end{bmatrix}$$

in terms of the basis $\left(\frac{\partial}{\partial x}(p), \frac{\partial}{\partial y}(p)\right)$ for $T_p(M)$.

Def. 5.8 A critical point $p \in M$ is **nondegenerate** if $\det(H(p)) \neq 0$. Otherwise, it is **degenerate**.

Def. 5.9 A smooth map $h: M \to \mathbb{R}$ is a **Morse function** if all its critical points are non-degenerate. (sometimes also require distinct function values)

Aside: Everything we have stated generalizes naturally to higher dimensions.

Lemma 5.1 (Morse Lemma)

It is possible to choose local coordinates x, y at crit. pt. $p \in M$ so that a Morse function h takes the form

$$h(x,y) = \pm x^2 \pm y^2$$

If $h(x,y) = x^2 + y^2$, then p is a minimum — index $= 0$

$\begin{cases} h(x,y) = x^2 - y^2 \\ h(x,y) = -x^2 + y^2 \end{cases}$, then p is a saddle pt — index $= 1$

$h(x,y) = -x^2 - y^2$, then p is a maximum — index $= 2$

Implication is that nondegenerate critical pts are isolated.

Def. 5.10 The **index** $i(p)$ of h at critical pt $p \in M$ is the number of negative eigenvalues of the Hessian $H(p)$.

Stable and unstable manifolds

Def. 5.12 Let γ be any curve passing through p, tangent to $v_p \in T_p(M)$.

The gradient ∇h of a Morse function h is

$$\frac{d\gamma}{dt} \cdot \nabla h = \frac{d(h \circ \gamma)}{dt}.$$

In general, can replace inner product with arbitrary Riemannian metric.

Note: directional derivative $v_p[h] = v_p \cdot \nabla h(p)$.

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Also, choosing coordinates x, y so that tangent vectors $\frac{\partial}{\partial x}(p)$, $\frac{\partial}{\partial y}(p)$ are orthonormal gives normal $\nabla h = \left(\frac{\partial h}{\partial x}(p), \frac{\partial h}{\partial y}(p) \right)$

The gradient of a morse function h is a vector field on $\mathbb{M}$.

We integrate this vector field to decompose $\mathbb{M}$ into regions of uniform flow.

Def. 5.13 An **integral line** $\gamma : \mathbb{R} \to \mathbb{M}$ is a maximal path whose tangent vectors agree with the gradient, i.e. $\frac{\partial}{\partial s}\gamma(s) = \nabla h(\gamma(s)) \quad \forall s \in \mathbb{R}$.

origin $\quad \text{org}\, \gamma = \lim_{s \to -\infty} \gamma(s)$

destination $\quad \text{dest}\, \gamma = \lim_{s \to +\infty} \gamma(s)$

Integral lines are open on both ends and the limits exist, and are crit. pts.

Thm 5.2 Integral lines have the following properties

(a) Two integral lines are either disjoint or the same

(b) Integral lines cover all of $\mathbb{M}$.

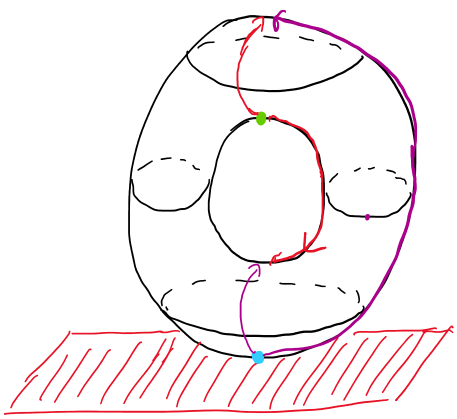
(c) The limits $\text{org}\, p$ and $\text{dest}\, p$ are crit. pts. of h.

Integral lines are just the paths following the gradient flow on the manifold.

Def. 5.14 The **stable manifold** $S(p) = \{p\} \cup \{y \in \mathbb{M} \mid y \in \text{im}\, \gamma,\ \text{dest}\, \gamma = p\}$

The **unstable manifold** $U(p) = \{p\} \cup \{y \in \mathbb{M} \mid y \in \text{im}\, \gamma,\ \text{org}\, \gamma = p\}$

where $\text{im}\, \gamma$ is the image of the path $\gamma : \mathbb{R} \to \mathbb{M}$.



Note that both sets of manifolds decompose $\mathbb{M}$ into open cells.

Def. 5.15 An open d-cell σ is a space homeomorphic to $\mathbb{R}^d$.

Thm 5.3 The stable manifold $S(p)$ of a critical pt p with index $i = i(p)$ is an open cell of dimension $\dim S(p) = i$.

Corollary: The unstable manifold $U(p)$ of a critical pt p with index $i = i(p)$ is an open cell of dimension $\dim U(p) = 2 - i$ (or $d - i$)

Note: Stable manifolds are pairwise disjoint because every pt can only approach a single critical pt in the limit, by following gradient flow. (Following ODE existence/uniqueness)

Aside: Note that closures of manifolds are NOT necessarily homeomorphic to a closed ball.

Stable manifolds decompose $\mathbb{M}$ into open cells.
Unstable manifolds are a dual decomposition.

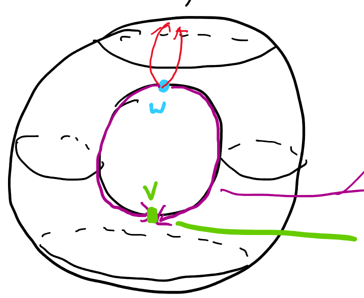
Let $a, b \in \mathbb{M}$ critical points. $\dim S(a) = 2 - \dim U(a)$

and $S(a) < S(b)$ iff $U(b) < U(a)$

$<$ is a face of

Ex. A minimum has a 2-cell as a stable manifold
a 0-cell as an unstable manifold.
A saddle pt has a 1-cell as a stable manifold
a 1-cell as an unstable manifold
A maximum has a 0-cell as a stable manifold
2-cell as an unstable manifold.

However, stable manifolds do not necessarily form a complex because it is possible that the boundary is not the union of lower-dim stable manifolds.



Def. 5.16 A Morse function is a **Morse-Smale function** if the stable and unstable manifolds intersect only transversally.

Transversality implies that the intersection of a stable q-manifold and an unstable p-manifold has dimension $q + p - d$

Also, the boundary of a stable manifold is a union of stable manifolds of lower dimension.
In 2D, stable and unstable 1-manifolds cross when they intersect.

Also, the boundary of a stable manifold is a union of stable manifolds of lower dimension.

In 2D, stable and unstable 1-manifolds cross when they happens to be at saddle pt. intersect,

Most generic possible intersections.

Def. 5.17 Connected components of sets $U(p) \cap S(q)$ for all critical points $p, q \in \mathbb{M}$ are **Morse-Smale cells**.

cell dim $= 0 \iff$ vertices
cell dim $= 1 \iff$ arcs
cell dim $= 2 \iff$ regions.

The collection of Morse-Smale cells forms a **Morse-Smale complex**.

Floer homology

Def. Let K be a Morse-Smale complex. An **q-chain** is a formal sum of index-q critical points $\sum c_q \sigma_q$, where $c_q \in \mathbb{F}$ and the sum is taken over all possible index-q critical pts $\sigma_q \in K$. The set of all q-chains is denoted $C_q(K)$.

Often, we let $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$

The **boundary** of an index-q crit. pt., u, is the sum of the index $q-1$ critical points connected to u by an edge in the Morse-Smale complex. If there are multiple edges, we add the index $q-1$ pt multiple times.

Chain complex

$$\cdots \xrightarrow{\partial} C_{q+1}(K) \xrightarrow{\partial} C_q(K) \longrightarrow C_{q-1}(K) \longrightarrow \cdots$$

We can of course define homology groups and Betti numbers the same way.

Morse inequalities Let $\mathbb{M}$ be a manifold of dimension d and $f: \mathbb{M} \to \mathbb{R}$ a Morse function.

Let $c_q = |\{\text{critical points of index } q\}|$, the number of such.

Then
(i) WEAK: $c_q \geq \beta_q(\mathbb{M})$ for all q
(ii) STRONG: $\sum_{q=0}^{j} (-1)^{j-q} c_q \geq \sum_{q=0}^{j} (-1)^{j-q} \beta_q(\mathbb{M})$ for all j.

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(iii) EQUALITY: $\sum_{q=0}^{d}(-1)^{d-q}c_q = \sum_{q=0}^{d}(-1)^{d-q}\beta_q(M)$.

Alternating sum all all numbers of critical pts of each index is the Euler characteristic.

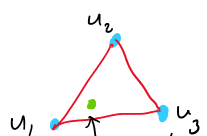
Possible to show relationship between Floer and singular homology.

Piecewise Linear Functions

Smooth functions are seldom found in practical situations, so we need to extend our Morse theory to at least some non-smooth functions.

Lower star filtration

Let K be a simplicial complex with real values at all vertices.
We can linearly extend to get a piece-wise linear (PL) function $f: |K| \to \mathbb{R}$ defined by $f(x) = \sum_i b_i(x) f(u_i)$, where u_i are vertices of K and $b_i(x)$ are barycentric coordinates of x



value there is the weighted sum

Assume f is generic, so can order vertices $f(u_1) < f(u_2) < \cdots < f(u_n)$.
(distinct function values)

Then we can define a subcomplex K_i by taking just the first i vertices.

i.e. $\sigma \in K_i$ iff $\sigma \in K$ and $\forall u_j \in \sigma$, $j \le i$.

Define: The **star** of a vertex u_i is the set of cofaces of u_i in K
$$St\, u_i = \{\sigma \in K \mid u_i \in \sigma\}$$

Define The **lower star** of a vertex u_i is the subset of simplices for which u_i is the vertex with maximum function value.
$$St_-\, u_i = \{\sigma \in St\, u_i \mid x \in \sigma \Rightarrow f(x) \le f(u_i)\}$$

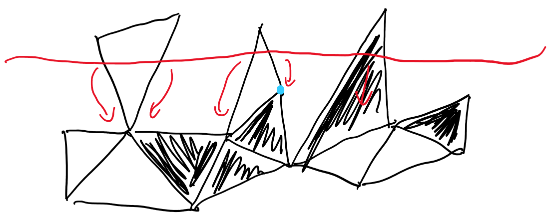
By genericity, each simplex has a unique maximum vertex, and thus belongs to a unique lower star.
$\Rightarrow$ lower stars partition K

And K_i is a union of the first i lower stars.

Define: The lower star filtration as the nested sequence of complexes

$$\emptyset \subseteq K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = K.$$

Notice: K_i are representative of the sublevel sets $|K|_a = f^{-1}(-\infty, a]$ as if $f(u_i) \leq a < f(u_{i+1})$, then $|K|_a$ and K_i have the same homotopy type.



can shrink down partial vertices

Define: The closed star $\overline{St}\, u_i$ of vertex u_i in a simplicial complex K is the smallest subcomplex of K that contains $St\, u_i$.

Define: the link of a vertex is the set of simplices in the closed star that are not in the star.

Define: The lower link is the collection of simplices in the closed lower star that are not in the lower star.

When going from K_{i-1} to K_i, we attach the closed lower star of u_i, gluing it along the lower link to the complex K_{i-1}.

If the lower link of u_i is non-empty but homologically trivial, then u_i is a PL regular vertex.

If the lower link of u_i has the reduced homology of a $(q-1)$-sphere, then u_i is a simple PL-critical vertex of index q

Define A piecewise-linear function $f: |K| \to \mathbb{R}$ on a manifold is a PL Morse function if each vertex is either PL regular or simple PL-critical, and the function values of the vertices are distinct.

We can the generalize our ideas from Morse functions & their critical pts to PL Morse functions in this way.

Euler characteristic: $\chi(K) = \sum_u (-1)^{\text{index}(u)}$ alternating sum of simple PL critical points

PL Morse inequalities: Let K be the triangulation of a manifold of dimension d, and $f: |K| \to \mathbb{R}$ a PL Morse function

PL Morse inequalities: Let K be the triangulation of a manifold of dimension d, and $f:|K|\to\mathbb{R}$ a PL Morse function.

Let $c_q = \#$ of index q PL critical points of f.

Then all the Morse inequalities above hold.