Problem Set 5

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Problem 1 [BHK 8.5-8.6] (30 points). Let $G(n, \frac{1}{n})$ be an Erdös-Renyi graph with n nodes and edge probability $\frac{1}{n}$.

1. Argue with high probability that there is no vertex of degree greater than $\frac{6 \log n}{\log \log n}$ (i.e. the probability that such a vertex exists goes to zero as $n \to \infty$). You may use the Poisson approximation and may wish to use the fact that $k! \geq \left(\frac{k}{e}\right)^k$

2. Prove that there is almost surely a vertex of degree $\Omega(\log n / \log \log n)$. See 8.1.1 for the outline of the argument; you will need to apply a technical fix for the problem that degrees of vertices are not independent.

Problem 2 [BHK 8.12] (40 points). Carry out an argument, similar to the one used for triangles, to show that $p = \frac{1}{n^{2/3}}$ is a threshold for the existence of a 4-clique. A 4-clique consists of four vertices with all (4 choose 2) edges present.

Problem 3 [BHK 8.14] (30 points). Let x be an integer chosen uniformly at random from $\{1, 2, \ldots, n\}$. Count the number of distinct prime factors of n. The exercise is to show that the number of prime factors almost surely is $\theta(\ln \ln n)$. Let p stand for a prime number between 2 and n.

1. For each fixed prime p, let I_p be the indicator function of the event that p divides x. Show that $\mathbb{E}(I_p) = \frac{1}{p} + O\left(\frac{1}{n}\right)$.

2. The random variable of interest, $y = \sum_p I_p$, is the number of prime divisors of x picked at random. Show that the variance of y is $O(\ln \ln n)$. For this, assume the known result that the number of primes p between 2 and n is $O(n / \ln n)$ and that $\sum_p \frac{1}{p} \approx \ln \ln n$. To bound the variance of y, think of what $\mathbb{E}(I_p I_q)$ is for $p \neq q$, both primes.

3. Use (1) and (2) to prove that the number of prime factors is almost surely $\theta(\ln \ln n)$.