Problem Set 6

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Problem 1 [Steif 2.3 & 2.5] (40 points). Consider percolation on $\mathbb{Z}^2$. We start with the graph $\mathbb{Z}^2$ which has vertices being the set $\mathbb{Z}^2$ and edges between pairs of points at Euclidean distance 1. Given $p \in [0, 1]$ We will construct a random subgraph of $\mathbb{Z}^2$ by letting each edge be *open* with probability p and *closed* with probability $1 - p$. Define $\theta(p)$ as the probability that the size of the open component containing the origin is infinite in size.

1. Show that $\theta(p)$ cannot be 1 for any $p < 1$.
2. Prove that $\theta(p) > 0$ for $p > \frac{2}{3}$.

Hint (part 2): Choose N so that $\sum_{n>N}^{\infty} n4(3^{n-1})(1-p)^n < 1$. Let E_1 be the event that all edges are open in $[-N, N] \times [-N, N]$ and E_2 be the event that there are no simple cycles in the dual surrounding $[-N, N]^2$ consisting of all closed edges. Look now at $E_1 \cap E_2$.

Problem 2 [BHK 11.1, 11.4] (30 points).

1. Give a solution to the dilation equation $f(x) = f(2x) + f(2x - k)$ satisfying $f(0) = 1$. Assume k is an integer.
2. What is the solution to the dilation equation

$$f(x) = f(2x) + f(2x - 1) + f(2x - 2) + f(2x - 3)$$

Problem 3 (30 points). We only proved results about the critical value for bond percolation on the square lattice. Write a program to numerically approximate the critical value on several other percolation models:

1. Site percolation on the square lattice.
2. Bond percolation on the triangular lattice.
3. Site percolation on the triangular lattice.
4. Bond percolation on the hexagonal lattice.
5. Site percolation on the hexagonal lattice.