Problem Set 7

[Your name] and [student ID]
MAT1850-2020 (Prof. Yun William Yu)

Problem 1 [Vol 2; 3.2] (5 points). Let $f : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be the function defined by

$$f(A) = A^3.$$

Prove that

$$Df_A(H) = A^2H + AHA + HA^2$$

for all $A, H \in \mathbb{R}^{n\times n}$.

Problem 2 [Vol 2: 3.4] (10 points). Recall that $\mathfrak{so}(n)$ denotes the vector space of real skew symmetric $n \times n$ matrices ($B^T = -B$). Let $C : \mathfrak{so}(n) \to \mathbb{R}^{n\times n}$ be the function given by

$$C(B) = (I - B)(I + B)^{-1}$$

1. ~~Prove that if B is skew-symmetric, then $I - B$ and $I + B$ are invertible, and so C is well-defined.~~ It has come to my attention that this is a repeat of a previous homework problem.

2. Prove that

$$dC(B)(A) = -[I + (I - B)(I + B)^{-1}]A(I + B)^{-1} = -2(I + B)^{-1}A(I + B)^{-1}$$

3. Prove that $dC(B)$ is injective for every skew-symmetric matrix B.

Problem 3 [Vol 2: 3.6] (20 points).

1. Consider the function g defined for all $A \in \mathbf{GL}(n, \mathbb{R})$, that is, all $n \times n$ real invertible matrices, given by

$$g(A) = \det(A).$$

   Prove that

$$dg_A(X) = \det(A)\operatorname{tr}(A^{-1}X)$$

   for all $n \times n$ real matrices X.

2. Consider the function f defined for all $A \in \mathbf{GL}^+(n, \mathbb{R}$, that is, $n \times n$ real invertible matrices of positive determinants, given by

$$f(A) = \log g(A) = \log\det(A).$$

   Prove that

$$df_A(X) = \operatorname{tr}(A^{-1}X)$$
$$D^2f(A)(X_1, X_2) = -\operatorname{tr}(A^{-1}X_1A^{-1}X_2),$$

   for all $n \times n$ real matrices X, X_1, X_2.

Problem 4 [Vol 2: 4.1-4.2] (10 points).

1. Find the extrema of the function $J(v_1, v_2) = v_2^2$ on the subset U given by

$$U = \{(v_1, v_2) \in \mathbb{R}^2 \mid v_1^2 + v_2^2 - 1 = 0\}$$

2. Find the extrema of the function $J(v_1, v_2) = v_1 + (v_2 - 1)^2$ on the subset U given by

$$U = \{(v_1, v_2) \in \mathbb{R}^2 \mid v_1^2 = 0\}$$

Problem 5 [Vol 2: 4.3] (20 points). Let A be an $n \times n$ real symmetric matrix, B an $n \times n$ symmetric positive definite matrix, and let $b \in \mathbb{R}^n$.

1. Prove that a necessary condition for the function J given by

$$J(v) = \frac{1}{2} v^\intercal A v - b^\intercal v$$

to have an extremum in $u \in U$, with U defined by

$$U = \{v \in \mathbb{R}^n \mid v^\intercal B v = 1\}$$

is that there is some $\lambda \in \mathbb{R}$ such that

$$Au - b = \lambda B u.$$

2. Prove that for all $(u, \lambda) \in U \times \mathbb{R}$, if $Au - b = \lambda Bu$, then

$$J(v) - J(u) = \frac{1}{2}(v - u)^\intercal (A - \lambda B)(v - u)$$

for all $v \in U$. Can you conclude that u is an extremum of J on U? Explain your reasoning.

Problem 6 [Vol 2: 4.4] (10 points). Let E be a normed vector space, and let U be a subset of E such that for all $u, v \in U$, we have $\frac{u+v}{2} \in U$.

1. Prove that if U is closed, then U is convex.
Hint. Every real $\theta \in (0, 1)$ can be written in a unique way as

$$\theta = \sum_{n \geq 1} \alpha_n 2^{-n},$$

with $\alpha_n \in \{0, 1\}$.

2. Does the result in (1) hold if U is not closed?

Problem 7 [Vol 2: 4.7-.48] (15 points).

1. Prove that the function $x \mapsto |x|^p$ is convex on $\mathbb{R}$ for all $p \geq 1$.
2. Prove that the function $x \mapsto \log x$ is concave on $\{x \in \mathbb{R} \mid x > 0\}$
3. Prove that the function $x \mapsto x \log x$ is convex on $\{x \in \mathbb{R} \mid x > 0\}$
4. Prove that the function f given by $f(x_1, \ldots, x_n) = \max\{x_1, \ldots, x_n\}$ is convex on $\mathbb{R}$.
5. Prove that the function g given by $g(x_1, \ldots, x_n) = \log(e^{x_1} + \cdots + e^{x_n})$ is convex on $\mathbb{R}$.

Problem 8 [Vol 2: 4.9] (10 points). You may wish to refer back to Problem 3 [Vol 2: 3.6]. Let $f : \mathbf{GL}^+(n, \mathbb{R}) \to \mathbb{R}$ be given by $f(A) = \log \det(A)$. Assume that A is symmetric positive definite, and let X be a symmetric matrix.

1. Prove that the eigenvalues of $A^{-1}X$ are real (even though $A^{-1}X$ may not be symmetric).
2. Prove that the eigenvalues of $(A^{-1}X)^2$ are nonnegative. Deduce that

$$D^2 f(A)(X, X) = -\operatorname{tr}\left((A^{-1}X)^2\right) < 0$$

for all nonzero symmetric matrices X and SPD matrices A. Conclude that the function $X \mapsto \log \det X$ is strictly concave on the set of symmetric positive definite matrices.