Problem Set 8

[Your name] and [student ID]
MAT1850-2020 (Prof. Yun William Yu)

Problem 1 [Vol 2; 5.1] (20 points). If $\alpha > 0$ and $f(x) = x^2 - \alpha$, Newton's method yields the sequence

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{\alpha}{x_k}\right)$$

to compute the square root $\sqrt{\alpha}$ of α.

(1) Prove that if $x_0 > 0$, then $x_k > 0$ and

$$\begin{aligned} x_{k+1} - \sqrt{\alpha} &= \frac{1}{2x_k}(x_k - \sqrt{\alpha})^2 \\ x_{k+2} - x_{k+1} &= \frac{1}{2x_{k+1}}(\alpha - x_{k+1}^2) \end{aligned}$$

for all $k \geq 0$. Deduce that Newton's method converges to $\sqrt{\alpha}$ for any $x_0 > 0$.

(2) Prove that if $x_0 < 0$, then Newton's method converges to $-\sqrt{\alpha}$.

Problem 2 [Vol 2; 5.2] (20 points).

(1) If $\alpha > 0$ and $f(x) = x^2 - \alpha$, show that the conditions of Theorem 5.1 are satisfied by any $\beta \in (0, 1)$ and any x_0 such that

$$|x_0^2 - \alpha| \leq \frac{4\beta(1-\beta)}{(\beta+2)^2} x_0^2$$

with

$$r = \frac{\beta}{\beta+2} x_0, \qquad M = \frac{\beta+2}{4x_0}.$$

(2) Prove that the maximum of the function defined on $[0, 1]$ by

$$\beta \mapsto \frac{4\beta(1-\beta)}{(\beta+2)^2}$$

has a maximum for $\beta = 2/5$. For this value of β, check that $r = x_0/6$, $M = 3/(5x_0)$, and

$$\frac{6}{7}\alpha \leq x_0^2 \leq \frac{6}{5}\alpha.$$

Problem 3 [Vol2; 5.4] (15 points).

(1) Show that Newton's method applied to the matrix function

$$f(X) = A - X^{-1}$$

with A and X invertible $n \times n$ matrices and started with any $n \times n$ matrix X_0 yields the sequence (X_k) with

$$X_{k+1} = X_k(2I - AX_k), \quad k \geq 0.$$

(2) If we let $R_k = I - AX_k$, prove that

$$R_{k+1} = I - (I - R_k)(I + R_k) = R_k^2$$

for all $k \geq 0$. Use this to prove that Newton's method converges to A^{-1} iff the spectral radius of $I - X_0A$ is strictly smaller than 1, that is, $\rho(I - X_0A) < 1$.

Problem 4 [Vol 2; 6.1] (10 points). Consider the relation

$$A \succeq B$$

between any two $n \times n$ matrices (symmetric or not) iff $A - B$ is symmetric positive semidefinite. Prove that this relation is a partial order.

Problem 5 [Vol 2; 6.3] (5 points). Find the minimum of the function

$$Q(x_1, x_2) = \frac{1}{2}\left(2x_1^2 + x_2^2\right)$$

subject to the constraint $x_1 - x_2 = 3$.

Problem 6 [Vol 2; 6.4] (10 points). Consider the problem of minimizing the function

$$f(x) = \frac{1}{2}x^\intercal Ax - x^\intercal b$$

in the case where we add an affine constraint of the form $C^\intercal x = t$, with $t \in \mathbb{R}^m$ and $t \neq 0$, and where C is an $n \times m$ matrix of rank $m \leq n$.

Give the details of the reduction of this constrained minimization problem to an unconstrained minimization problem. *Hint:* see Section 6.2, and the reduction for linear constraints of the form $C^\intercal x = 0$.

Problem 7 [Vol 2; 6.5] (10 points). Find the maximum and minimum of the function

$$Q(x, y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

on the unit circle $x^2 + y^2 = 1$.

Problem 8 [Vol 2; 12.1] (10 points). Let V be a Hilbert space. Prove that a subspace W of V is dense in V if and only if there is no nonzero vector orthogonal to W.