Problem Set 9

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This problem set is based on Vol 2, problem 13.3.

Let A be a real $n \times n$ symmetric positive definite matrix and let $b \in \mathbb{R}^n$.

Problem 1 [25pts]. Prove that if we apply the steepest descent method (for the Euclidean norm) to

$$J(v) = \frac{1}{2}\langle Av, v\rangle - \langle b, v\rangle,$$

and if we define the norm $||v||_A$ by

$$||v||_A = \langle Av, v\rangle^{1/2},$$

we get the inequality

$$||u_{k+1} - u||_A^2 \leq ||u_k - u||_A^2 \left(1 - \frac{||A(u_k - u)||_2^4}{||u_k - u||_A^2 ||A(u_k - u)||_A^2}\right).$$

Problem 2 [25pts]. Recall that the condition number $\text{cond}_2(A) = \lambda_n/\lambda_1$, where λ_n is the largest eigenvalue of A and λ_1 is the smallest eigenvalue of A.

Consider the case where $n = 2$. Prove that

$$||u_{k+1} - u||_A \leq \frac{\text{cond}_2(A) - 1}{\text{cond}_2(A) + 1}||u_k - u||_A,$$

and thus

$$||u_k - u||_A \leq \left(\frac{\text{cond}_2(A) - 1}{\text{cond}_2(A) + 1}\right)^k ||u_0 - u||_A.$$

Problem 3 [25pts]. Let's return to the general case of $n \geq 2$. Using a diagonalization of A, where the eigenvalues of A are denoted $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, prove that

$$||u_{k+1} - u||_A \leq \frac{\text{cond}_2(A) - 1}{\text{cond}_2(A) + 1}||u_k - u||_A.$$

Note that this proof can be rather tricky. *Hint*: you may find it helpful to normalize something to a unit vector, to consider the convexity of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(\lambda) = \frac{1}{\lambda}$, and to bound something by a linear approximation.

Problem 4 [25pts]. Prove that when $\text{cond}_2(A) = 1$, then $A = \lambda I$ for some $\lambda \in \mathbb{R}$ and the method converges in one step. Further prove that when the initial starting descent direction $Au_0 - b$ is an eigenvector of A, the method also converges in one step.