What is a number?
Counting, addition, & subtraction
Lecture 1a: 2022-01-10

MAT A02 – Winter 2022 – UTSC

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Who are you?

- What are you studying? (type “a”, “b”, “c”, “d”, or “e” in chat)

  A: Arts, literature, and language
  B: History, philosophy, and cultural studies
  C: Social and behavioral sciences
  D: Something not listed above
  E: Undecided

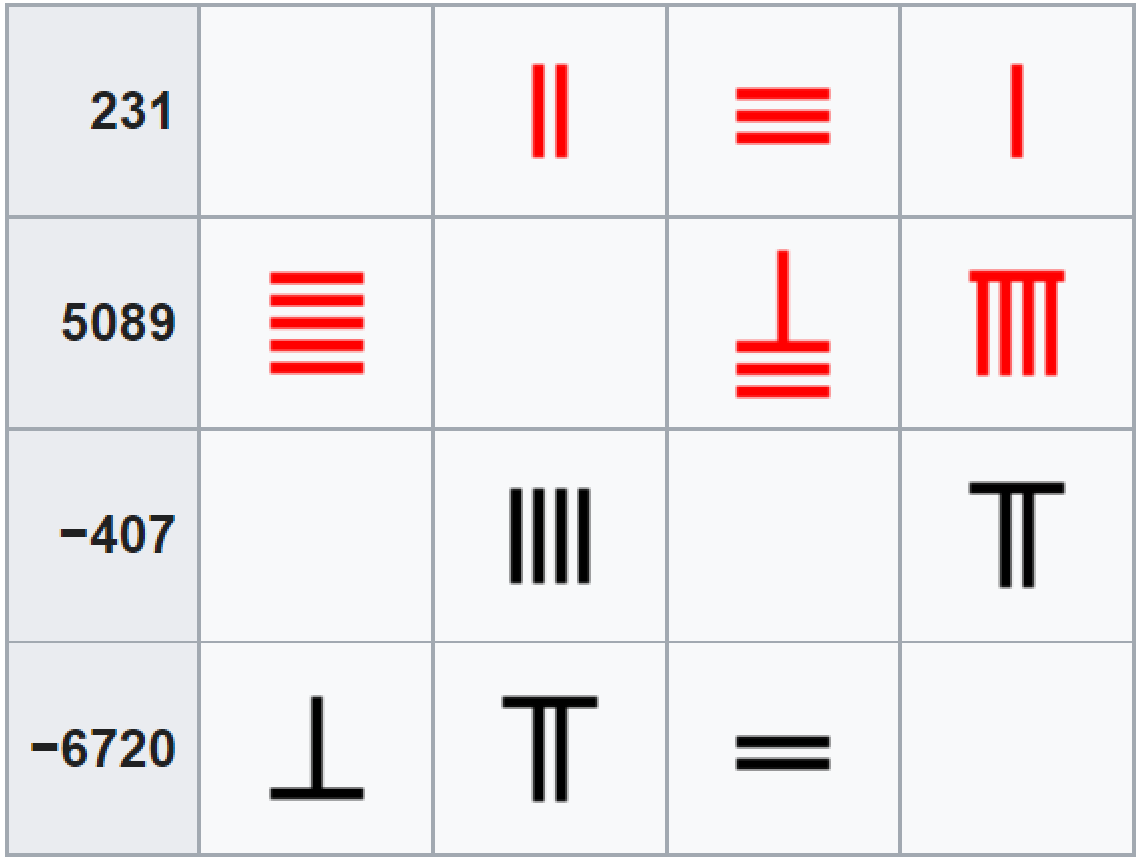
- What year of university study are you in?

  A: 1st
  B: 2nd
  C: 3rd
  D: 4th
  E: 5+

- You may also add “?” to pump up the confusion meter.

When were negative numbers invented?

A: Before 1000 BCE
B: 1000 BCE to 1000 CE
C: 1000 CE to 1500 CE
D: 1500 CE to 1800 CE
E: After 1800 CE



Chinese counting rods, circa 202 BCE – 220 CE
https://en.wikipedia.org/wiki/Counting_rods

Questions to explore in MATA02

- What is a number?
  - Relationship to counting and measurements
  - Common operations on numbers (addition, subtraction, multiplication, division, exponentiation, roots)
- Can we extend what it means to be a number?
  - Clock arithmetic (modular arithmetic)
  - Real and complex numbers
- What’s so special about prime numbers?
  - How many are there?
  - Can we find where they are?
- How are prime numbers used in our everyday lives?
  - RSA encryption (used for online security “https”)

Natural numbers (counting numbers)

- 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, …

- “0” is a late addition to the natural numbers, since it took mathematicians a lot longer to figure out that it needed a name.
- Are negative numbers “natural”?
- Are fractions “natural”?
- Are imaginary numbers “natural”?

A: Yes
B: No
C: Maybe???
D: Mathematicians are silly and come up with weird arbitrary definitions.
E: None of the above

How to invent addition

- When putting together groups of objects, counting is slow

Think like a mathematician

- Have I seen this problem before?
  - (more formally, prove that a new problem can be reframed as an old problem you already know how to solve)
- Once you have reduced a problem to a previously solved problem, your job as a mathematician is done.
- Counting circles and counting squares is the same, so you might consider creating a table that let's you look up putting together two numbers:

Mathematical notation: + and =

- Saying that we want to group together one group of 3 objects and another group of 5 objects to get a group of 8 objects is tedious. Let’s invent symbols.
- Plus sign + used as $x + y$, where x and y are arbitrary numbers, means that we are counting the number total number of objects when we group together a group with x objects and another group with y objects.
- Equal sign = is used to denote that two expressions are the same

Addition properties

- Commutative property: $x + y = y + x$

- Associative property: $(x + y) + z = x + (y + z)$

- Additive identity property: $x + 0 = x$

- Distributive property (later when we invent multiplication)

How to invent subtraction

- What happens when we take away items?

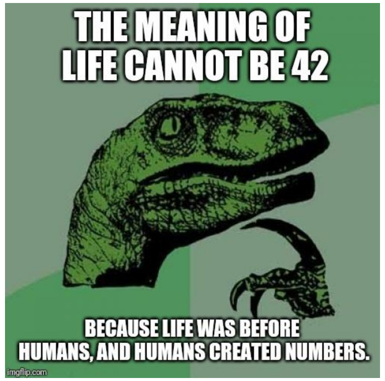
- We could create a table like we did for addition.

- But then the table is incomplete because some operations don't give an answer (are "undefined")

What should we do?

A: It’s fine. We don’t need all subtractions to make sense.

B: Let’s invent more numbers!



C: All of math is pointless

Inventing negative numbers

- What if we double all of the natural numbers except 0 and put a minus sign in front of the copies?

- The left copy of the numbers we refer to as "negative numbers"
- Subtraction $x - y$ is well-defined when $x > y$.
- Let's define $y - x$ where $x > y$ to be equal to $-(x - y)$

Think like a mathematician

- What problem remains after having invented negative numbers?

A: We don’t know how to subtract negative numbers
B: We don’t know how to add negative numbers
C: We don’t know how addition and subtraction interact
D: All of the above
E: None of the above

The number line

- Let’s write the negative and positive numbers on a long line, with negative numbers to the left and positive numbers to the right:
- Another way to understand addition of positive numbers is how far to the right we are moving along the number line.
- Another way to understand addition of negative numbers is by moving to the left on the number line.
- Subtraction means to move in the opposite direction, or to add the negative of a number

Teaser for future lectures

- We will invent multiplication, division, and square roots for the “integers” (i.e. all positive and negative whole numbers) the same kind of way.
- Notice that we made a choice to invent negative numbers though. What if instead of making a copy of the numbers, we turn the number line into a number circle? This will be the basis for “clock arithmetic” or “modular arithmetic”.