Inventing division
Lecture 1d: 2022-01-12

MAT A02 – Winter 2022 – UTSC

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Think like a mathematician

- Mathematicians have a toolkit of problems they've solved already, and they try to turn a new problem into one they've seen before.
- Consider measuring the length of a rope by assigning a number on the positive number line.
- What lengths do we still not know how to compute?

A: Cutting the rope in half
B: Joining together two ropes
C: Joining together 1000 ropes
D: Cutting a piece of known length off the rope, and measuring the remainder.
E: None of the above

Inventing division

- Subtraction allows you to "reverse" addition

$5 + 3 = 8$ $\qquad$ $8 - 3 = 5$

- We can define division "÷" as "reversing" multiplication

$5 \times 3 = 15$ $\qquad$ $15 \div 3 = 5$

- Definition: if $x \times y = z$, then $z \div y = x$
- What about $z \div y$ when there exists no x such that $x \times y = z$?

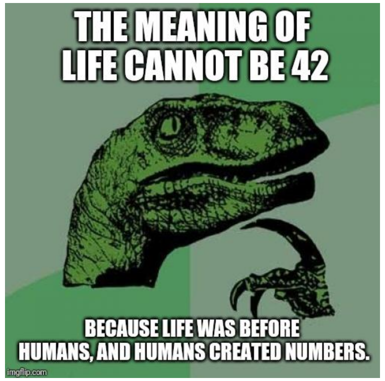
e.g. $5 \div 2$ $\qquad$ $2 \times 2 = 4$

$2 \times 3 = 6$

What should we do?

A: It’s fine. We don’t need all divisions to make sense.

B: Let’s invent more numbers!



C: All of math is pointless

D: It’s fine. The answer doesn’t need to be a number.

Inventing fractions

- What if we simply define a new number by $x \div y = \frac{x}{y}$ for all integers (positive or negative whole numbers) x and y?

Please type in chat what goes wrong

- Numbers now have multiple labels

$\frac{4}{2} = 2$ or $\underbrace{\frac{3}{1} = 3}_{\text{new names for old numbers}}$ or $\overbrace{\frac{2}{5} = \frac{4}{10}}^{\text{new number}}$

- Definition is inconsistent if $y = 0$, so we don't allow it.

$\frac{3}{0}$ implies $3 \div 0 = \frac{3}{0}$ so $\frac{3}{0} \times 0 = 3$

but 0 times anything is 0

need calculus and limits to formalize ∞

Equivalence of fractions

- When are two fractions different ways of writing the same number?

$$\frac{x}{y} = \frac{z}{w}$$

$$\frac{x}{y}\cdot y = \frac{z}{w}\cdot y$$

$$x = \frac{z}{w}\cdot y$$

$$x \cdot w = z \cdot y$$

$$\frac{2}{5} = \frac{4}{10} \qquad \frac{2}{5} = \underbrace{\cancel{\tfrac{2}{2}}}_{1}\cdot\frac{2}{5}$$

$$2\cdot 10 = 5\cdot 4 = 20$$

Also, $\frac{0}{5} = \frac{0}{3}$

- When the numerator of one fraction times the denominator of the other is equal to its denominator ties the numerator of the other.

times

Characterizing reciprocals

- With the invention of negative numbers, we can also "reverse" addition by adding a negative number.

$5 + 3 = 8 \qquad 8 + (-3) = 5$

- Once we've invented fractions, we can also reverse multiplication by any $x \neq 0$ by multiplying by $\frac{1}{x}$ (the "reciprocal")

$5 \times 3 = 15 \qquad 15 \div 3 = 5$

$\frac{15}{3} = 15 \times \frac{1}{3} = 5$

Fractions and addition

- How do we add together fractions? We can use the distributive property and reciprocals.

$$2 + \frac{5}{3} = 2 \cdot 1 + \frac{5}{3} = 2 \cdot \frac{3}{3} + \frac{5}{3}$$

$$= 2 \cdot 3 \cdot \frac{1}{3} + 5 \cdot \frac{1}{3} = \frac{2 \cdot 3}{3} + \frac{5}{3}$$

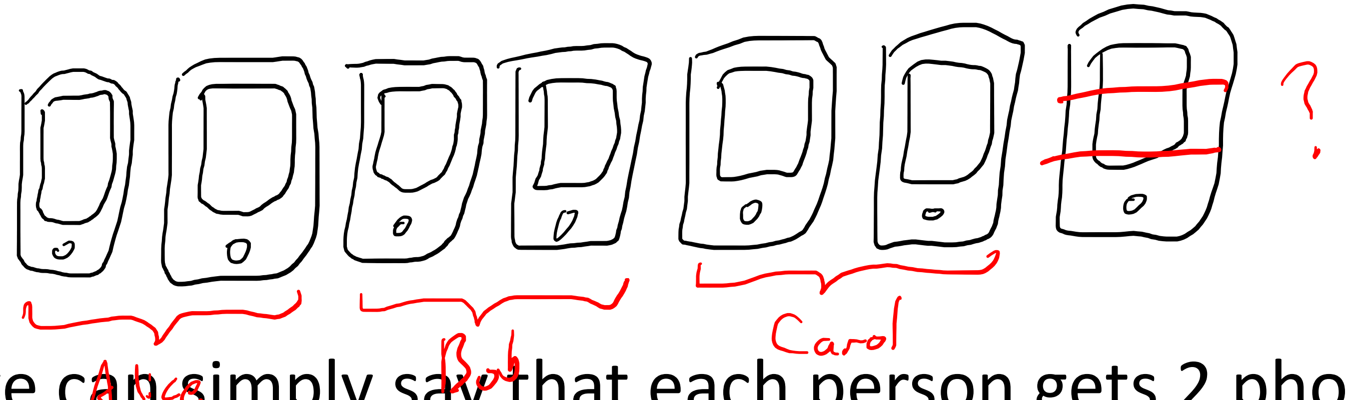
distributive prop

$$= (2 \cdot 3 + 5) \cdot \frac{1}{3} = (6 + 5) \cdot \frac{1}{3}$$

$$= 11 \cdot \frac{1}{3} = \frac{11}{3}$$

Alternative: division with remainder

- Alternately, we can have our answer not be a single number.
- Consider dividing 7 phones among 3 people evenly. Using fractions is a bad idea.



- Instead, we can simply say that each person gets 2 phones, with 1 phone "remaining" as the "remainder".

$7 \div 3 = 2$ remainder 1

or $7 = 2 \cdot 3 + 1$

division is the reverse

Connection between two divisions

- Division with remainder and division giving fractions can be converted into each other because if we convert the fraction to an integer plus a fraction where the numerator is greater than the denominator, the remainder is the numerator.

$$7 \div 3 = \frac{7}{3} = \frac{6+1}{3} = \frac{6}{3} + \frac{1}{3} = 2 + \frac{1}{3}$$

$7 \div 3 = 2$ remainder 1

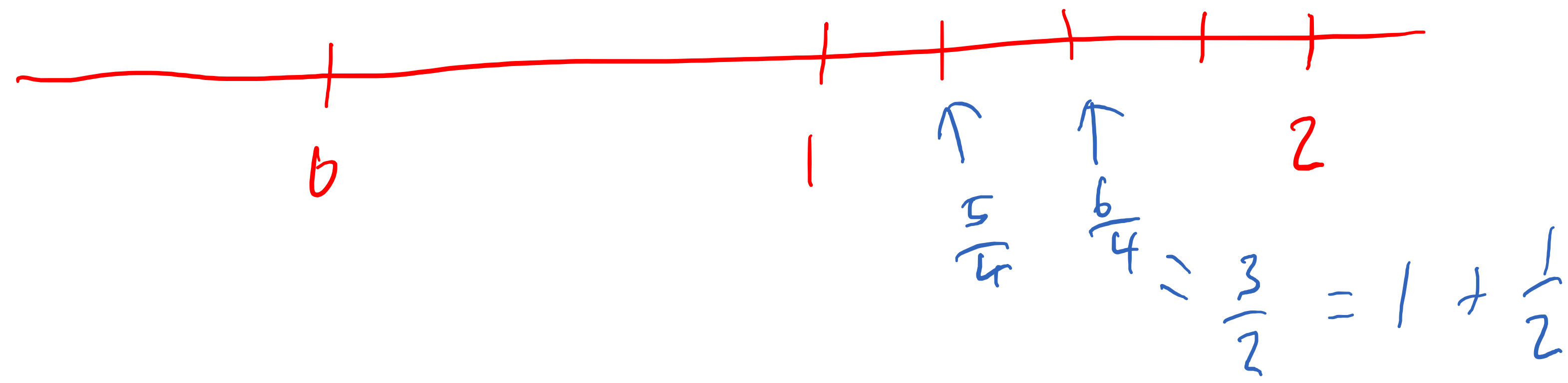
$$7 = 2 \cdot 3 + 1$$

$$\frac{7}{3} = 2 + \frac{1}{3}$$

Putting fractions on the number line

- Using the form from the previous slide, we can put fractions on the number line by using the remainder and even spacing.

$\frac{7}{4} = 1 + \frac{3}{4}$ ← 3rd tick ← 4 even space



- Any number we can write as a fraction is a “rational number”

Try it out: solve a linear equation

- $\frac{1}{2}x - \frac{3}{8} = \frac{3x}{10} + \frac{1}{8}$

$\frac{1}{2}x = \frac{3x}{10} + \frac{1}{8} + \frac{3}{8}$ add $\frac{3}{8}$

$\frac{1}{2}x = \frac{3x}{10} + \frac{4}{8}$

(multiply 8)

$4x = \frac{24x}{10} + 4$

$4x = \frac{12x}{5} + 4$

$4x - \frac{12x}{5} = 4$ (subtract $\frac{12x}{5}$)

(multiply 5)

$20x - 12x = 20$

$8x = 20$

(divide 8) $x = \frac{20}{8} = \frac{5}{2}$

$\frac{1}{8} + \frac{3}{8} = 1 \cdot \frac{1}{8} + 3 \cdot \frac{1}{8}$

$= (1+3) \cdot \frac{1}{8}$

$= 4 \cdot \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$

A: $x = -\frac{5}{4}$

B: $x = \frac{5}{2}$

C: $x = \frac{5}{4}$

D: $x = 5$

E: None of the above

On decimals

- Finite decimal numbers can be thought of as a fraction over 1, 10, 100, 1000, 10,000, and so on (powers of 10, but we haven't invented powers yet).

$$1.415 = \frac{1415}{1000} \quad \text{or} \quad 1 + \frac{4}{10} + \frac{1}{100} + \frac{5}{1000}$$

- Most fractions do not give a finite decimal number.

$$\frac{1}{3} = 0.3333\ldots$$

- Properly understanding infinite decimals requires understanding limits, which is rigorously done in calculus, though we can approximate with a finite decimal.

$$\frac{1}{7} \approx 0.142857$$