Inventing division
Lecture 1d: 2022-01-12

MAT A02 – Winter 2022 – UTSC

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Think like a mathematician

- Mathematicians have a toolkit of problems they've solved already, and they try to turn a new problem into one they've seen before.
- Consider measuring the length of a rope by assigning a number on the positive number line.
- What lengths do we still not know how to compute?

A: Cutting the rope in half
B: Joining together two ropes
C: Joining together 1000 ropes
D: Cutting a piece of known length off the rope, and measuring the remainder.
E: None of the above

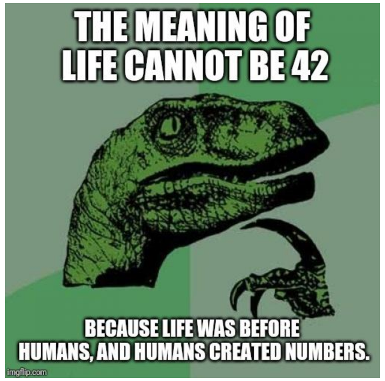
Inventing division

- Subtraction allows you to “reverse” addition
- We can define division “÷” as “reversing” multiplication
- Definition: if $x \times y = z$, then $z \div y = x$
- What about $z \div y$ when there exists no x such that $x \times y = z$?

What should we do?

A: It's fine. We don't need all divisions to make sense.

B: Let's invent more numbers!



C: All of math is pointless

D: It's fine. The answer doesn't need to be a number.

Inventing fractions

- What if we simply define a new number by $x \div y = \frac{x}{y}$ for all integers (positive or negative whole numbers) x and y?

Please type in chat what goes wrong

- Numbers now have multiple labels

- Definition is inconsistent if $y = 0$, so we don't allow it.

Equivalence of fractions

- When are two fractions different ways of writing the same number?

- When the numerator of one fraction times the denominator of the other is equal to its denominator ties the numerator of the other.

Characterizing reciprocals

- With the invention of negative numbers, we can also "reverse" addition by adding a negative number.

- Once we've invented fractions, we can also reverse multiplication by any $x \neq 0$ by multiplying by $\frac{1}{x}$ (the "reciprocal")

Fractions and addition

- How do we add together fractions? We can use the distributive property and reciprocals.

Alternative: division with remainder

- Alternately, we can have our answer not be a single number.
- Consider dividing 7 phones among 3 people evenly. Using fractions is a bad idea.

- Instead, we can simply say that each person gets 2 phones, with 1 phone “remaining” as the “remainder”.

Connection between two divisions

- Division with remainder and division giving fractions can be converted into each other because if we convert the fraction to an integer plus a fraction where the numerator is greater than the denominator, the remainder is the numerator.

Putting fractions on the number line

- Using the form from the previous slide, we can put fractions on the number line by using the remainder and even spacing.

- Any number we can write as a fraction is a "rational number"

Try it out: solve a linear equation

- $\frac{1}{2}x - \frac{3}{8} = \frac{3x}{10} + \frac{1}{8}$

A: $x = -\frac{5}{4}$
B: $x = \frac{5}{2}$
C: $x = \frac{5}{4}$
D: $x = 5$
E: None of the above

On decimals

- Finite decimal numbers can be thought of as a fraction over 1, 10, 100, 1000, 10,000, and so on (powers of 10, but we haven’t invented powers yet).

- Most fractions do not give a finite decimal number.

- Properly understanding infinite decimals requires understanding limits, which is rigorously done in calculus, though we can approximate with a finite decimal.