Powers and Roots
Lecture 2a: 2022-01-17

MAT A02 – Winter 2022 – UTSC

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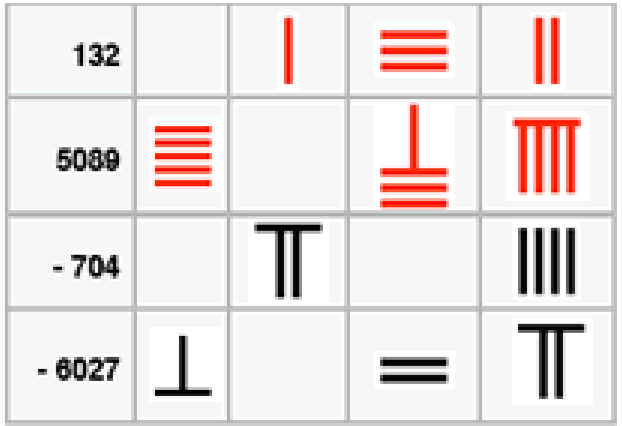
Different reasons to invent numbers

- Practical applicability
  - Negative numbers made it easier to talk about financial debt.
  - Fractions let you measure smaller quantities of a continuous substance
- Theoretical consistency
  - Negative numbers mean that all subtractions have an answer.
  - Fractions mean that almost all divisions have an answer.

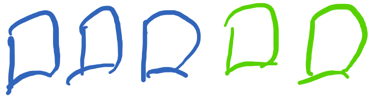


$\frac{5}{0}$ is still not allowed

Inventing operations

negative numbers

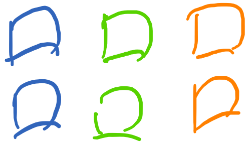
- Repeated counting = addition. Reversed by subtraction.



$$3 + \overbrace{1 + 1}^{\text{count twice after } 3} = 5 \qquad 5 - 3 = 2$$

fractions

- Repeated addition = multiplication. Reversed by division.



$$3 \times 2 = \overbrace{2 + 2 + 2}^{3 \text{ copies}} = 6 \qquad 6 \div 2 = \frac{6}{2} = 3$$

- Repeating multiplication = powers/exponents

$$2^3 = \underbrace{2 \times 2 \times 2}_{3 \text{ copies}} = 8$$

reversal

$$8 \xrightarrow{?} 2$$

$$\sqrt[3]{8} = 2$$

Think like a mathematician

- In school, we learn about addition, multiplication, and exponentiation. Can we keep on going?

A: Yes
B: No
C: Maybe
D: Why bother?
E: None of the above

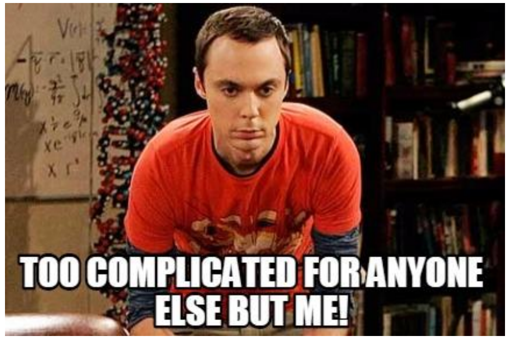
$2^{2^{2}} = 2 \uparrow\uparrow 3$ "tetration"

Repeated "tetration" = "pentation"

"hexation", etc.

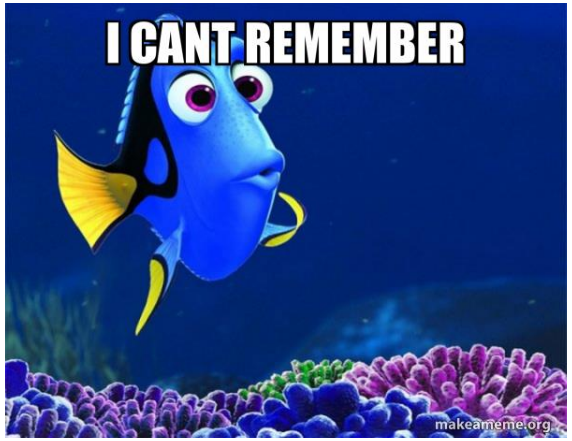
- Why don't we learn about these more in school?

A: Too complicated



B: Practically useless

C: We did! Don't you remember?



Why care about powers?

- Suppose you owe $10,000 in student debt which grows at 10% a year. How much do you owe in 10 years if you don't pay it off?

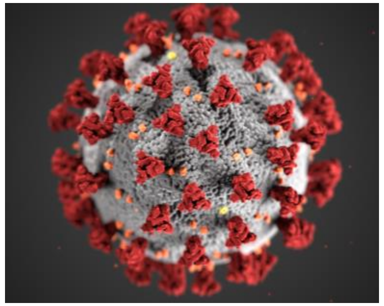
$10{,}000 \times (1.1) \times (1.1) \cdots \times (1.1)$

$= 10{,}000 \times (1.1)^{10} \approx \cancel{\$61{,}159}$

$\$25{,}937$

A: $10,000
B: $20,000
C: $30,000
D: $40,000
E: Even more money

- Suppose the number of people infected by COVID doubles every 3 days. If one person is infected today, about how many are infected after a month?



$30 \rightarrow \frac{30}{3} = 10$ doublings

$1 \times 2 \times 2 \cdots \times 2 = 1 \cdot 2^{10} = 1024$

A: 10
B: 100
C: 1,000
D: 10,000
E: Even more people

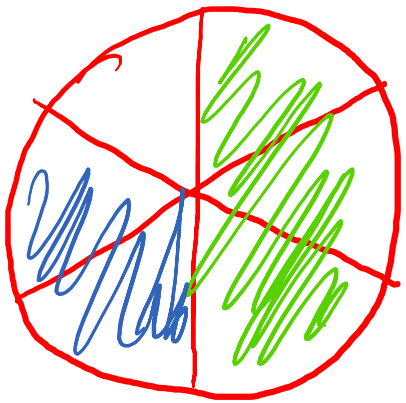
Relationships to existing numbers

- Inventing numbers and operations isn't enough; we need to figure out how they relate to existing numbers/operations.
- Ex. distributive rule: $a(x+y) = ax + ay$

$$2(5+x) = 10 + 2x$$

properties & theorems

- Subtle consequence of the distributive rule: adding fractions



$$\frac{1}{3}+\frac{1}{2} \neq \frac{2}{5}$$

$$= \frac{5}{6}$$

$$\frac{1}{3}+\frac{1}{2} = 1\cdot\frac{1}{3} + 1\cdot\frac{1}{2}$$

$$= \frac{2}{2}\cdot\frac{1}{3} + \frac{3}{3}\cdot\frac{1}{2} = \frac{2}{6}+\frac{3}{6}$$

$$= \underbrace{2\cdot\left(\frac{1}{6}\right)+3\cdot\left(\frac{1}{6}\right)}_{\text{distributive}} = (2+3)\cdot\frac{1}{6}$$

$$= 5\cdot\frac{1}{6} = \frac{5}{6}$$

Exponents and addition

- $\overbrace{a^{m+n} = a^m a^n}$, m, n positive integers

$$2^{3+4} = \underbrace{2\cdot 2\cdot 2}_{3\text{ times}}\cdot\underbrace{2\cdot 2\cdot 2\cdot 2}_{4\text{ times}}$$

$$= 2^3 \cdot 2^4$$

proof. Notice $a^{m+n} = \underbrace{a\cdot a\cdots a}_{m+n\text{ times}}$

But, we can group the multiplications together into first m times and the next n times

$$a^{m+n} = \underbrace{a\cdot a\cdots a}_{m\text{ times}}\cdot\underbrace{a\cdot a\cdots a}_{n\text{ times}}$$

And by the definition of exponents,

$$a^{m+n} = a^m \cdot a^n$$

$\boxed{/\!/}$

Exponents and subtraction

- $a^{m-n} = \frac{a^m}{a^n}$ }

$$2^{4-2} = \frac{2^4}{2^2}$$

$$= \frac{2 \cdot 2 \cdot \cancel{2} \cdot \cancel{2}}{\cancel{2} \cdot \cancel{2}} = 2^2$$

Easy case: $m > n$, positive integers

Then $\frac{a^m}{a^n} = \frac{a \cdot a \cdot a \cdots a}{a \cdot a \cdots a}$ ← m times ← n times

So we can cancel out n of the a's on top + bottom
left with $m-n$ a's on top $= a^{m-n}$

In a rigorous math class, would check all
the other cases. What if $n > m$?

Negative exponents

- $a^{-m} = \dfrac{1}{a^m}$

$$a^n \cdot a^{-m} = a^{n+(-m)} = a^{n-m} = \frac{\boxed{a^n}}{a^m}$$

divide both sides by a^n

$$\Rightarrow \frac{a^n a^{-m}}{a^n} = \frac{a^n}{a^m} \cdot \frac{1}{a^n}$$

$$\Rightarrow a^{-m} = \frac{1}{a^m}$$

Ex. $2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$

0-exponents

- $a^0 = 1$ for any $a \neq 0$

$$a^0 = a^{1-1} = a^1 a^{-1} = \frac{a}{a} = 1$$

Ex.

$$\left(\log 5000 + e^{15} + \pi \cdot \sin 30\right)^0 = 1$$

Exponents and multiplication

- $(a^m)^n = a^{mn}$

$$(a^m)^n = \overbrace{a^m \cdot a^m \cdot \ldots \cdot a^m}^{n \text{ times}}$$

$$= a^{m+m \cdots + m}$$

$$= a^{m \cdot n}$$

$$(2^3)^4 = 2^3 \cdot 2^3 \cdot 2^3 \cdot 2^3$$

$$= 2^{3+3+3+3}$$

$$= 2^{3 \cdot 4} = 2^{12}$$

$$= 4096$$

Ex. $3^4 = (3^2)^2 = 9^2 = 81$

Multiplication and exponents

- $(ab)^m = a^m b^m$

$$(ab)^m = \underbrace{\underbrace{a \cdot b} \cdot \underbrace{a \cdot b} \cdots \underbrace{a \cdot b}}_{m \text{ times}} = \underbrace{a \cdots a}_{m \text{ times}} \cdot \underbrace{b \cdots b}_{m \text{ times}}$$

$$= a^m \cdot b^m$$

Ex: $6^2 = (2 \cdot 3)^2 = 2 \cdot 3 \cdot 2 \cdot 3 = 2 \cdot 2 \cdot 3 \cdot 3$

$= 2^2 \cdot 3^2 = 4 \cdot 9 = 36$

- Similar to distributive property $m(a+b) = ma + mb$ because just as multiplication is repeated addition, exponentiation is repeated multiplication.

Fractional exponents and roots

- $a^{\frac{1}{n}} = \sqrt[n]{a}$ the nth root of a.

$a^{\frac{1}{2}} = \sqrt{a}$

$a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1$

$\Rightarrow a^{\frac{1}{3}} = \sqrt[3]{a}$ cube root

<u>Ex.</u> $8^{\frac{1}{3}} = 2$ because $2 \cdot 2 \cdot 2 = 8$

Exponent rules summarized for $a, b > 0$

- $a^0 = 1$
- $(a^m)^n = a^{mn}$: taking a power multiplies exponents together
- $a^m a^n = a^{m+n}$: multiplying like powers adds the exponents
- $a^{-m} = \frac{1}{a^m}$: negative exponents are reciprocals
- $(ab)^m = a^m b^m$

$2^4 3^3 \neq 6^7$

Ex. $16^{-\frac{1}{4}} 3^2 = \frac{1}{16^{\frac{1}{4}}} \cdot 3^2 = \frac{3^2}{16^{\frac{1}{4}}} = \frac{9}{\left(16^{\frac{1}{2}}\right)^{\frac{1}{2}}} = \frac{9}{\sqrt{\sqrt{16}}}$

$= \frac{9}{\sqrt{4}} = \frac{9}{2}$

Try it out

- $\left(81^{\frac{1}{4}}\right)^{3} \cdot 2^{-3}2^{2}$

A: $\frac{27}{2}$
B: 1
C: $\frac{27}{64}$
D: $\frac{64}{27}$
E: None of the above

- $\left(\left(81^{\frac{1}{4}}\right)^{6} \cdot 2^{-4}2^{3}\right)^{0}$

Think like a mathematician

- We already have a lot of different kinds of numbers: natural numbers, negative numbers, and fractions.
- Do we need even more kinds of numbers for powers?
- What about for roots?

A: Yes
B: No
C: Maybe
D: Too many numbers!
E: None of the above