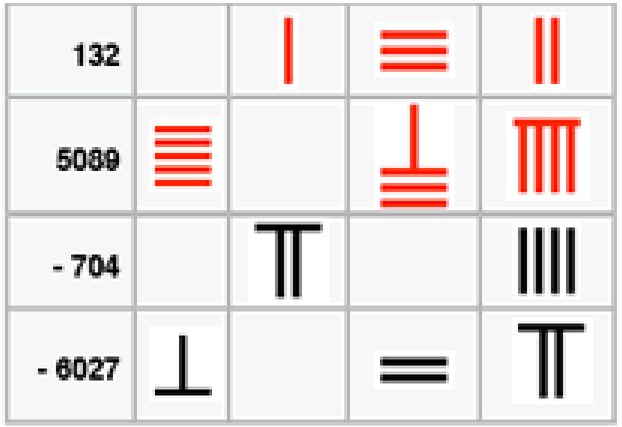
Powers and Roots
Lecture 2a: 2022-01-17

MAT A02 – Winter 2022 – UTSC

Prof. Yun William Yu

Different reasons to invent numbers

- Practical applicability
  - Negative numbers made it easier to talk about financial debt.
  - Fractions let you measure smaller quantities of a continuous substance
- Theoretical consistency
  - Negative numbers mean that all subtractions have an answer.
  - Fractions mean that almost all divisions have an answer.



Inventing operations

- Repeated counting = addition. Reversed by subtraction.

- Repeated addition = multiplication. Reversed by division.

- Repeating multiplication = powers/exponents

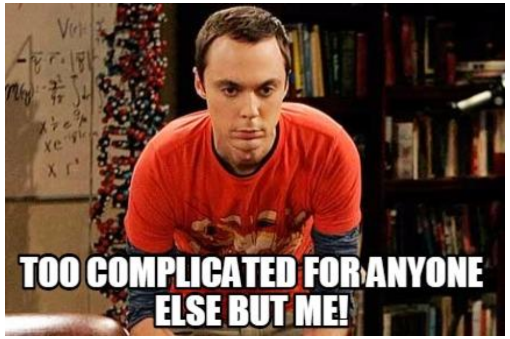
Think like a mathematician

- In school, we learn about addition, multiplication, and exponentiation. Can we keep on going?

A: Yes
B: No
C: Maybe
D: Why bother?
E: None of the above

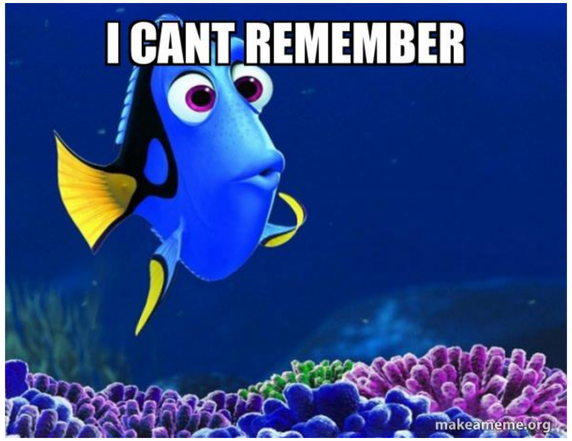
- Why don't we learn about these more in school?

A: Too complicated



B: Practically useless

C: We did! Don't you remember?

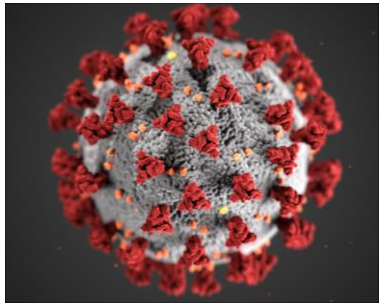


Why care about powers?

- Suppose you owe $10,000 in student debt which grows at 10% a year. How much do you owe in 10 years if you don't pay it off?

A: $10,000
B: $20,000
C: $30,000
D: $40,000
E: Even more money

- Suppose the number of people infected by COVID doubles every 3 days. If one person is infected today, about how many are infected after a month?



A: 10
B: 100
C: 1,000
D: 10,000
E: Even more people

Relationships to existing numbers

- Inventing numbers and operations isn't enough; we need to figure out how they relate to existing numbers/operations.
- Ex. distributive rule: $a(x + y) = ax + ay$

- Subtle consequence of the distributive rule: adding fractions

Exponents and addition

- $a^{m+n} = a^m a^n$

Exponents and subtraction

- $a^{m-n} = \frac{a^m}{a^n}$

Negative exponents

- $a^{-m} = \frac{1}{a^m}$

0-exponents

- $a^0 = 1$ for any $a \neq 0$

Exponents and multiplication

- $(a^m)^n = a^{mn}$

Multiplication and exponents

- $(ab)^m = a^m b^m$

- Similar to distributive property $m(a+b) = ma + mb$ because just as multiplication is repeated addition, exponentiation is repeated multiplication.

Fractional exponents and roots

- $a^{\frac{1}{n}} = \sqrt[n]{a}$ the nth root of a.

Exponent rules summarized for $a, b > 0$

- $a^0 = 1$
- $(a^m)^n = a^{mn}$: taking a power multiplies exponents together
- $a^m a^n = a^{m+n}$: multiplying like powers adds the exponents
- $a^{-m} = \frac{1}{a^m}$: negative exponents are reciprocals
- $(ab)^m = a^m b^m$

Try it out

- $\left(81^{\frac{1}{4}}\right)^3 \cdot 2^{-3}2^2$

- $\left(\left(81^{\frac{1}{4}}\right)^6 \cdot 2^{-4}2^3\right)^0$

A: $\frac{27}{2}$
B: 1
C: $\frac{27}{64}$
D: $\frac{64}{27}$
E: None of the above

Think like a mathematician

- We already have a lot of different kinds of numbers: natural numbers, negative numbers, and fractions.
- Do we need even more kinds of numbers for powers?

A: Yes
B: No
C: Maybe
D: Too many numbers!
E: None of the above

- What about for roots?