How to solve it
Lecture 2b: 2022-01-19

MAT A02 – Winter 2022 – UTSC

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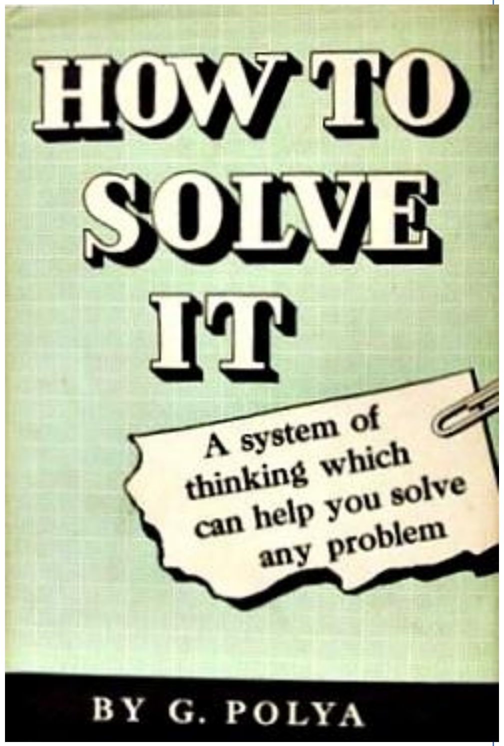
Codifying solution strategies

- How to Solve It (1945) by George Polya
  - Understand the problem
    - What’s the start and end point of the problem?
    - Can you rephrase the problem?
    - Can you draw a picture?
  - Devise a plan
    - What methods have you learned in class?
    - Are there special cases that are easier?
    - Can you work backwards?
  - Carry out the plan
    - Often what you’ve practice, but you have to be careful.
  - Look back on your work
    - Did you check your work using an alternate strategy?
    - What strategies worked for you on this problem, and can you use it on similar future problems?



Understanding the problem

- Solve for x in the following: $\frac{3}{7}x + \frac{2}{35} = \frac{5}{3}x - \frac{14}{15}$
- What's the start point?

A: An equation with two variables
B: An equation with fractions
C: Facts about the current state of the world
D: A set of known operations
E: None of the above

fractions are the biggest part

- What's the end point?

A: Knowing the value of x
B: An interpretation of the equation
C: A prediction for the future
D: A newly invented kind of number
E: None of the above

want to know

$x = ?$

Devise a plan: $\frac{3}{7}x + \frac{2}{35} = \frac{5}{3}x - \frac{14}{15}$

- What are methods you learned for manipulating equations?

Respond in chat

- Replacing a value with something equal to that value.
- Additive/multiplicative identities.
- Doing the same thing to both sides.
- Commutative/Associative/Distributive properties.
- How to add fractions

$x+2=\frac{4}{2}$

$\Rightarrow x+2=2$ because $\frac{4}{2}=2$

$\frac{2}{2}\cdot x+\frac{x}{2}+\frac{3}{2}=0$

$2x+x+3=0$

$2(x+1)=4x+2$

$\frac{1}{3}x+\frac{1}{2}x=\frac{2}{6}x+\frac{3}{6}x=\frac{5}{6}x$

Execute the plan: $\frac{3}{7}x + \frac{2}{35} = \frac{5}{3}x - \frac{14}{15}$

- Use manipulations to convert to $x = ?$

Multiply everything by 35:

$$35\left(\frac{3}{7}x + \frac{2}{35}\right) = 35\left(\frac{5}{3}x - \frac{14}{15}\right)$$

$$15x + 2 = \frac{165}{3}x - \frac{7 \cdot \cancel{5} \cdot 14}{3 \cdot \cancel{5}}$$

$$15x + 2 = \frac{165}{3}x - \frac{98}{3}$$

Multiply everything by 3

$$45x + 6 = 165x - 98$$

Subtract both sides by 45x

$$6 = 120x - 98$$

Add 98 to both sides

$$104 = 120x$$

$$x = \frac{104}{120}$$

$$x = \frac{52}{60} = \frac{26}{30} = \frac{13}{15}$$

Look back: $\frac{3}{7}x + \frac{2}{35} = \frac{5}{3}x - \frac{14}{15}$

- Did you check your work using an alternate strategy? $x = \frac{13}{15}$

Plug in potential solution:

$$\frac{\cancel{3}}{7}\cdot\frac{13}{\underset{5}{\cancel{15}}} + \frac{2}{35} = \frac{\cancel{5}}{3}\cdot\frac{13}{\underset{3}{\cancel{15}}} - \frac{14}{15}$$

$$\frac{13}{35} + \frac{2}{35} = \frac{13}{9} - \frac{14}{15}$$

$$\frac{15}{35} = \frac{13}{9} - \frac{14}{15}$$

$$\frac{3}{7} = \frac{13}{9} - \frac{14}{15}$$

$$\frac{45}{7} = \frac{65}{3} - 14$$

$$6\frac{3}{7} = 21\frac{2}{3} - 14$$

$$6\frac{3}{7} = 7\frac{2}{3}$$

Look back

$35 = 7 \cdot 5$

$$\begin{array}{r} 35 \\ \times\ 5 \\ \hline 25 \\ 15 \\ \hline 175 \end{array}$$

$7 \times 5 = 35$

- Why did we not get the right solution?

A: We did
B: Wrong strategy
C: Error in execution
D: Bad assumptions
E: None of the above

Multiply by 35

$$35 \cdot \left(\frac{3}{7}x + \frac{2}{35}\right) = 35\left(\frac{5}{3}x - \frac{14}{15}\right)$$

$$15x + 2 = \frac{175}{3}x - \overbrace{7 \cdot 8}^{35} \cdot \frac{14}{3 \cdot 8}$$

$$15x + 2 = \frac{175x}{3} - \frac{98}{3}$$

$$45x + 6 = 175x - 98$$

$$6 = 130x - 98$$

$$130x = 104$$

$$x = \frac{104}{130} = \frac{52}{65} = \frac{13 \cdot 4}{13 \cdot 5} = \frac{4}{5}$$

$$35\left(\frac{5}{3}x - \frac{14}{15}\right) = 35 \cdot \frac{5}{3}x - \cancel{35}^{7} \cdot \frac{14}{\cancel{15}_{3}}$$

Recheck our new work

- Does $x = \frac{4}{5}$ solve $\frac{3}{7}x + \frac{2}{35} = \frac{5}{3}x - \frac{14}{15}$?

$$\frac{3}{7} \cdot \frac{4}{5} + \frac{2}{35} = \frac{5}{3} \cdot \frac{4}{5} - \frac{14}{15}$$

$$\frac{12}{35} + \frac{2}{35} = \frac{20}{15} - \frac{14}{15}$$

$$\frac{14}{35} = \frac{6}{15}$$

$$\frac{2 \cdot 7}{5 \cdot 7} = \frac{2 \cdot 3}{5 \cdot 3}$$

$$\frac{2}{5} = \frac{2}{5} \quad \checkmark$$

Look back and reflect

- What strategies worked on this problem; what can you use in the future?

  Respond in chat

- Multiplying both sides of the equation by the denominator to simplify the equation.
- Alternately, can multiply by $1 = \frac{m}{m}$, but that can be more complicated to keep track of.
- After getting rid of the denominator, “moving” everything with an x to one side, and “moving” everything else to the other side.
- Checking the final answer by plugging it in can help when you make a mistake.

Why care about powers?

$(1.1)^{10} = (1.1^5)^2$

- Suppose you owe \$10,000 in student debt which grows at 10% a year. How much do you owe in 10 years if you don't pay it off?

$10{,}000 \times (1.1) \times (1.1) \times \cdots \times (1.1)$

$= 10{,}000 \times (1.1)^{10} \approx \cancel{\$61{,}159}$

\$25,937

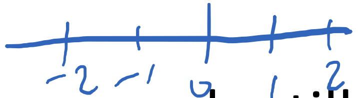
A: \$10,000
B: \$20,000
C: \$30,000
D: \$40,000
E: Even more money

- Understand the problem: need to predict the future value based on a yearly growth rate of 10%.
- Devise a strategy: multiply initial value by 1.1 ten times.
- Execute the plan: see above.
- Look back and check: try $(1.1^5)^2 = (1.61)^2 = 2.59$, so clearly there's a problem in our computation.

Inventing negative numbers

$3 - 5 \neq$ pos. number

- Understand the problem:
  - When you only have positive integers, sometimes subtracting things doesn't give a number.
- Devise a strategy:
  - Invent some new numbers, and give them a name: "negative" integers
  - Then need to create a new number system with both positive and negative integers that fits them all together.



  - Need to check that addition/subtraction both work properly still.
- Execute the strategy:
  - Actually do the math and create new addition/subtraction tables.

$3 - 5 = -2$

- Look back and reflect:
  - Does this solve the original problem? Can we use this kind of method on other problems in the future, like division and fractions?

Try it out

and it is a simplified form.

- Rewrite $\left(81^{\frac{1}{4}}\right)^3 \cdot 2^{-3}2^2$ as $\frac{m}{n}$, where m, n are both integers.
- Understand the problem: turn it into a fraction.
- Design a strategy: use exponentiation rules
  - $a^0 = 1$
  - $(a^m)^n = a^{mn}$: taking a power multiplies exponents together
  - $a^m a^n = a^{m+n}$: multiplying like powers adds the exponents
  - $a^{-m} = \frac{1}{a^m}$: negative exponents are reciprocals
  - $(ab)^m = a^m b^m$
- Implement the strategy
- Look back and reflect; can you check your answer?

Try it out

- $\left(\overset{3}{\overline{81^{\frac{1}{4}}}}\right)^3 \cdot 2^{-3}2^2$ $= 3^3 \cdot 2^{-3+2}$

  $\left(81^{\frac{1}{2}}\right)^{\frac{1}{2}} = 9^{\frac{1}{2}} = 3$

  $= 27 \cdot 2^{-1}$

  $= \frac{27}{2}$

- $\left(\left(81^{\frac{1}{4}}\right)^6 \cdot 2^{-4}2^3\right)^0$ $= 1$

A: $\frac{27}{2}$
B: 1
C: $\frac{27}{64}$
D: $\frac{64}{27}$
E: None of the above

Try it out

- Rewrite $\left(\frac{x^{-1}y^{2}z^{0}}{x^{3}y^{-4}z^{2}}\right)^{-1}$ as a product or quotient of powers in which each variable occurs but once, and all exponents are positive. Assume all variables represent positive real numbers only.
- Understand the problem:
  - Similar to previous problem, except instead of wanting integers at the end, we want positive powers of x, y, z.
- Design a strategy:
  - Use exponentiation rules to remove negative powers and cancel out.
- Look back and reflect: Execution!
  - Is there an easy way to check the answer?
  - How similar is this to solving the previous problem?

Try it out: $\left(\dfrac{x^{-1}y^2z^0}{x^3y^{-4}z^2}\right)^{-1}$ $= \dfrac{x^3y^{-4}z^2}{x^{-1}y^2z^0}$

$= \left(\dfrac{x^{-1}y^2}{x^3y^{-4}z^2}\right)^{-1}$ because $z^0=1$

$= \left(\dfrac{x}{x}\cdot\dfrac{y^4}{y^4}\cdot\dfrac{x^{-1}y^2}{x^3y^{-4}z^2}\right)^{-1}$ multiply by $1=\dfrac{x}{x}=\dfrac{y^4}{y^4}$

$= \left(\dfrac{y^6}{x^4z^2}\right)^{-1}$

$= \dfrac{x^4z^2}{y^6}$ reciprocal

A: $\dfrac{x^2}{y^2z^2}$

B: $\dfrac{x^4}{y^2z^2}$

C: $\dfrac{x^2y^6}{z^2}$

D: $\dfrac{x^4z^2}{y^6}$

E: None of the above

$x=1$ $y=2$ $z=3$

$\left(\dfrac{1\cdot 2^2\cdot 1}{1^3\cdot 2^{-4}\cdot 3^2}\right)^{-1} = \left(\dfrac{4}{\frac{1}{16}\cdot 9}\right)^{-1} = \left(\dfrac{64}{9}\right)^{-1}$

$= \dfrac{9}{64}$

$\dfrac{1^4\cdot 3^2}{2^6} = \dfrac{9}{64}$ ✓

Think like a mathematician

- We already have a lot of different kinds of numbers: natural numbers, negative numbers, and fractions.
- Do we need even more kinds of numbers for powers?

$2^{10} = 2 \times 2 \times 2 \cdots \times 2$ No

- What about for roots?

$\sqrt{2}$ is NOT a fraction

A: Yes
B: No
C: Maybe
D: Too many numbers!
E: None of the above