Combinations
Lecture 3a: 2022-01-24

MAT A02 – Winter 2022 – UTSC

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Let's bake a cake

- We're going to have to measure ingredients.

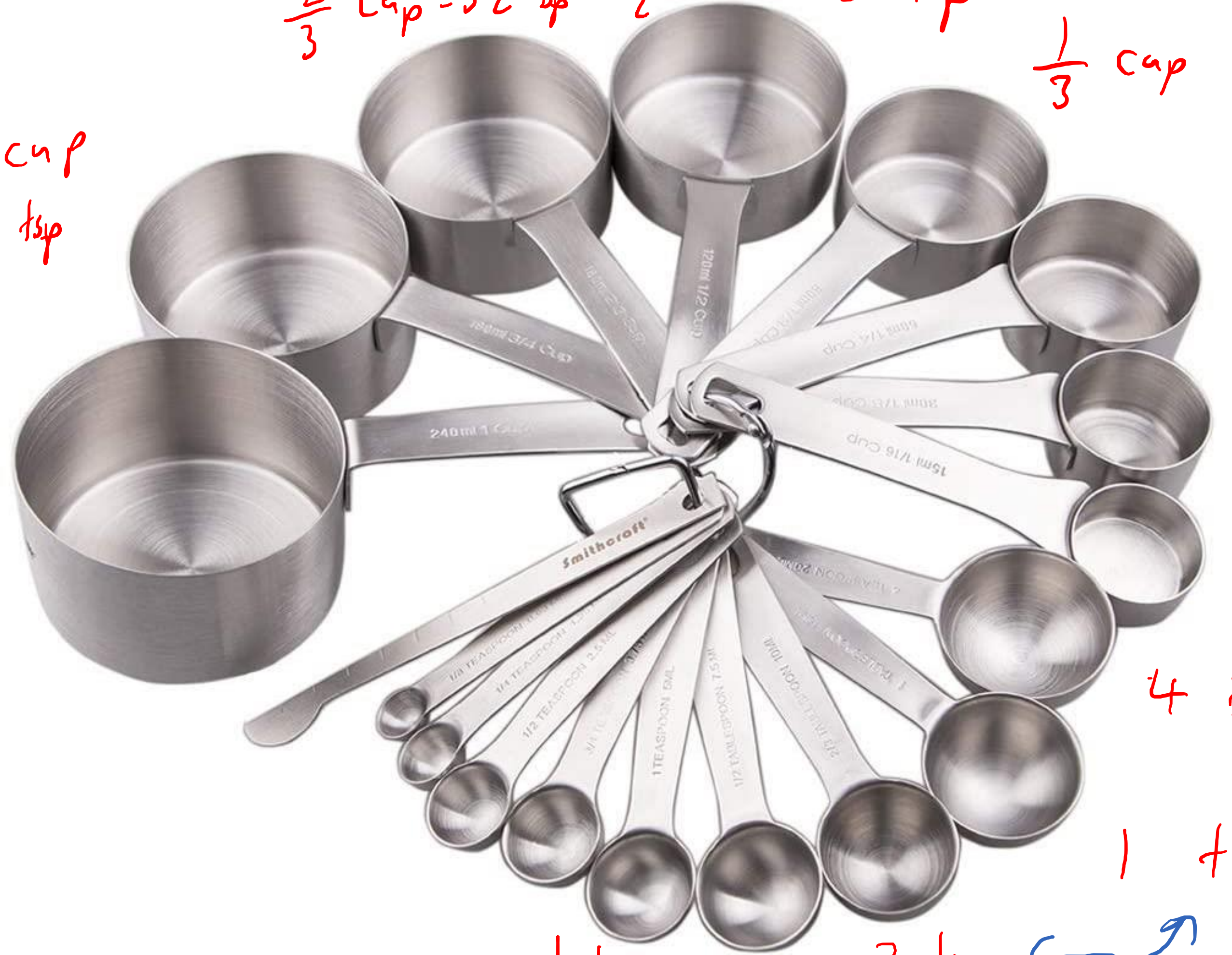
$\frac{2}{3}$ cup = 32 tsp

$\frac{1}{2}$ cup = 24 tsp

$\frac{1}{3}$ cup = 16 tsp

$\frac{3}{4}$ cup = 36 tsp

1 cup = 48 tsp



$\frac{1}{4}$ cup = 12 tsp

$\frac{1}{8}$ cup = 6 tsp

$\frac{1}{16}$ cup = 3 tsp

4 tsp

1 tbsp = 3 tsp

1 tsp

2 tsp

Can we measure the right amount?

- Let's say we want to measure 5 tsp cinnamon.
- Can we do it with a 2 tsp spoon and a 1tbsp = 3 tsp spoon?

Easy : 2 tsp + 3 tsp = 5 tsp

- What about 1tbsp = 3 tsp spoon and a third cup = 16 tsp cup?

Yes in theory, Start by measuring 2 third cups = 32 tsp

Remove 3 tsp , 9 times = -27 tsp

Left with 5 tsp

- What about a 4 tsp spoon and a 1 cup = 48 tsp cup?

No. Why? Even / odd

multiples of 4

A: Yes
B: No
C: Yes, but only in theory
E: None of the above

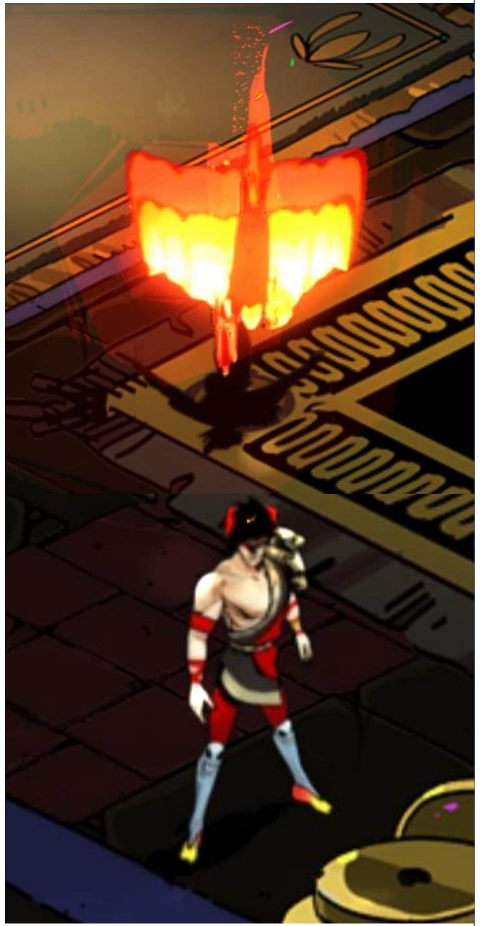
Let’s play a game

- In some video games, can move normally, or can do a fixed-length “dash”.
- When playing with a keyboard, a single button press may also go a specific distance.
- If the dash goes 9 ft, and walking goes 2 ft, can you go exactly 25 ft?

$\underbrace{9+9+9}_{27}-2=25$ $\quad$ $\underbrace{2+2+2+2+2+2+2+2}_{16}+9=25$

- What if you speed up walking to go 3 ft?

No. Can only go multiples of 3



A: Yes
B: No
C: Yes, but only in theory
E: None of the above

Combinations

- Given two integers a and b, the sum of a multiple of a and a multiple of b (allowing negative multiples) is called a combination of a and b.

$$a \cdot x + b \cdot y, \text{ where } x, y \text{ are integers.}$$

Ex. $a = 11$ $b = 7$ $11 \cdot 2 + 7 \cdot 3 = 43$ is a combo of 11 and 7

- Given two integers a and b, what numbers n are combinations of a and b?
- If n is a combination of a and b, how do we find multiples of a and b actually adding up to n?

$43 = 11x + 7y$. Find x & y as integers

Creating Unity

- Once you've figured out how to go 1 step, you could in theory figure out how to go any whole number of steps just by repeating the whole procedure.
- Example: 1tbsp = 3 tsp spoon and a third cup = 16 tsp cup

$$1 \text{ tsp} = 16 \text{ tsp} - 3 \text{ tsp} \times 5$$

- Then we just repeat that any number of times, showing that we can make every number as a combination.

$$5 \text{ tsp} = 5(16 \text{ tsp} - 3 \text{ tsp} \times 5)$$
$$= 16 \text{ tsp} \times 5 - (3 \text{ tsp} \times 5) \times 5$$
$$= 16 \text{ tsp} \times 5 - 3 \text{ tsp} \times 25$$
$$80 - 75$$

Earlier:

$$2 \times 16 - 3 \times 9$$
$$32 - 27$$

Process for making smaller combos?

- Let's keep on making smaller combinations, and then use those as building blocks.
- Ex. Combinations of 35 and 100.
- Can repeatedly subtract 35 from 100 to get a smaller combo.

  $100 - 35 = 65 \qquad 65 - 35 = 30$

- Can rewrite as division with remainder.

  $100 \div 35 = 2 \text{ r } 30 \qquad \text{so} \qquad 30 = 100 - 35 \times 2$

- Now we know how to make 30, so can use as a building block.

  $35 - 30 = 5 \qquad \text{so} \qquad 5 = 35 - (100 - 35 \times 2) = 35 \times 3 - 100$

- But now we're stuck at 5, since $30 \div 5 = 6$.

  $30 \to 25 \to 20 \to 15 \to 10 \to 5 \to 0$

The Euclidean algorithm and combinations

- If we start with two whole numbers a and b, then the smallest combo we can make uses the following algorithm:
- Divide a into b, and let r be the remainder.
  - If $r = 0$, then we're done; a divides b and $\gcd(a, b) = a$.
  - If $r \neq 0$, then we replace (a, b) with (r, a) and repeat.
- This is just the Euclidean algorithm, so the smallest combo we can make is precisely the greatest common divisor $\gcd(a, b)$.
- This means we know how to make any multiple of the $\gcd(a, b)$.

Ex 6 & 4

$6 \div 4 = 1 \text{ r } 2$

$4 \div 2 = 2$

$2 = \gcd(6, 4)$

Goal 10

$2 = 6 - 4$

$5 \times 2 = 5 \times 6 - 5 \times 4$

$10 = 5 \times 6 - 5 \times 4$

Note: Not the only way to make 10

6 + 4

Other combinations?

- Can we make any positive numbers that are not multiples of $\gcd(a, b)$?

No

A: Yes
B: No
C: Yes, but only in theory
E: None of the above

proof. Let $d = \gcd(a, b)$

Then $a = m \cdot d$ $b = n \cdot d$, where m, n are integers

Any combination is the $x \cdot a + y \cdot b$, x, y integers

$= x \cdot m \cdot d + y \cdot n \cdot d$

$= \underbrace{(x \cdot m + y \cdot n)}_{\text{whole number}} \cdot d$ ← a multiple of d. $\boxed{/\!/}$

- The set of combos of a, b is precisely the multiples of $\gcd(a, b)$

Try it out

- What is the set of combinations of 36 and 3222?

$3222 = 36 \times 89 + 18$

$36 = 18 \times 2$

$\gcd(3222, 36) = 18$

$$\begin{array}{r} 89 \; r\;18 \\ 36\,\overline{)\,3222} \\ \underline{288} \\ 342 \\ \underline{324} \\ 18 \end{array}$$

A: Multiples of 9
B: Multiples of 18
C: Multiples of 36
D: Multiples of 72
E: None of the above

Try it out (recall)

- Find the least common multiple of 36 and 3222?

$$a \times b = \gcd(a,b) \times \mathrm{lcm}(a,b)$$

$$\gcd(3222, 36) = 18$$

$$\mathrm{lcm}(36, 3222) = \frac{36 \times 3222}{18} = \frac{36}{18} \times 3222$$

$$= 2 \times 3222 = 6444$$

A: 3222
B: 6444
C: 9333
D: 12888
E: None of the above

Try it out

- Find integers x, y such that $36x + 3222y = 9$

Recall: $\gcd(36, 3222) = 18$

So only multiples of 18 are possible.

So 9 is NOT possible.

What if we guessed b?

$36 \times 89 + 3222 \neq 9$

A: $x = 178, y = 2$
B: $x = 89, y = 1$
C: $x = -89, y = 1$
D: $x = -178, y = 2$
E: None of the above

Try it out

- Find integers x, y such that $36x + 3222y = 18$

Recall:

$$\left.\begin{aligned} 3222 &= 36 \cdot 89 + 18 \\ 36 &= 18 \cdot 2 \end{aligned}\right\}$$ Euclid's Algorithm

$18 = 3222 - 36 \cdot 89$

$y = 1 \qquad x = -89$

A: $x = 178, y = 2$
B: $x = 89, y = 1$
C: $x = -89, y = 1$
D: $x = -178, y = 2$
E: None of the above

Try it out

- Find integers x, y such that $36x + 3222y = 36$

We know $18 = 3222 - 36 \times 89$

$\Rightarrow 36 = 3222 \times 2 - 36 \times 178$

$y = 2 \qquad x = -178$

A: $x = 178, y = 2$
B: $x = 89, y = 1$
C: $x = -89, y = 1$
D: $x = -178, y = 2$
E: None of the above

Aside: Another sol is

$x = 1, \quad y = 0$

Try it out

- Find integers x, y such that $36x + 3222y = 45$

Recall $\gcd(36, 3222) = 18$

$45 \div 18 = 2 \text{ r } 9$

not a multiple of 18, so not possible

A: $x = 178, y = 2$
B: $x = 89, y = 1$
C: $x = -89, y = 1$
D: $x = -178, y = 2$
E: None of the above

More complicated example

$133, 26$

$133 \div 26 = 5 \text{ r } 3$ $\quad 133 = 26 \cdot 5 + 3$ $\quad \Rightarrow \underline{3} = 133 - 26 \cdot 5$

$26 \div 3 = 8 \text{ r } 2$ $\quad 26 = 3 \cdot 8 + 2$ $\quad \Rightarrow 2 = 26 - 3 \cdot 8$

$3 \div 2 = 1 \text{ r } 1$ $\quad 3 = 2 \cdot 1 + 1$ $\quad \Rightarrow 1 = 3 - 2$

$2 \div 1 = 2$ $\quad 2 = 1 \cdot 2$

$\downarrow$

$\gcd(133, 26) = 1$

$1 = 3 - 2$

$1 = 3 - (26 - 3 \cdot 8)$

$1 = 3 \cdot 9 - 26$

$1 = (133 - 26 \cdot 5) \cdot 9 - 26$

$1 = 133 \cdot 9 - 26 \cdot 46$

$\rightarrow 3 = 133 \cdot 27 - 16 \cdot 138$

More complicated example

2022, 133

More complicated example

$2022, \ 133$

$$
\begin{array}{r}
15 \text{ r } 27 \\
133 \overline{)2022} \\
\underline{133} \\
692 \\
\underline{665} \\
27
\end{array}
$$

$$2022 = 133 \cdot 15 + 27$$
$$133 = 27 \cdot 4 + 25$$
$$27 = 25 \cdot 1 + 2$$
$$25 = 2 \cdot 12 + 1$$
$$2 = 1 \cdot 2$$

$$\gcd(133, 2022) = 1$$

$$1 = 25 - 2 \cdot 12$$
$$1 = 25 - (27 - 25) \cdot 12$$
$$1 = 25 \cdot 13 - 27 \cdot 12$$
$$1 = (133 - 27 \cdot 4) \cdot 13 - 27 \cdot 12$$
$$1 = 133 \cdot 13 - 27 \cdot 64$$
$$1 = 133 \cdot 13 - (2022 - 133 \cdot 15) \cdot 64$$
$$1 = 133 \cdot 973 - 2022 \cdot 64$$