Prime patterns
Lecture 4a: 2022-01-31

MAT A02 – Winter 2022 – UTSC

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Patterns in sets of numbers

- Natural numbers

  1, 2, 3, 4, 5, 6, 7, ... (via counting)

- Negative numbers

  -1, -2, -3, -4, -5, -6, -7, ... Formula $a_n = -n$

  $a_3 = -3$

- Even numbers

  2, 4, 6, 8, 10, 12, 14, ... Formula $a_n = 2n$

  $a_2 = 4$

  $a_3 = 6$

- Multiples of 3

  3, 6, 9, 12, 15, ... Formula $a_n = 3n$

- Odd numbers

  1, 3, 5, 7, 9, 11, ... Formula $a_n = 2n - 1$

  $a_1 = 1$ $a_2 = 3$

More complicated sets

- All positive integers less than or equal to 12

  $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

  $a_n = n$ for n from 1 to 12

- All even numbers between 19 and 31

  $\{20, 22, 24, 26, 28, 30\}$

  $a_n = 2n + 18$ for n from 1 to 6

- Perfect squares

  $1, 4, 9, 16, 25, 36, 49, \ldots$

  $a_n = n^2$

- Numbers that are divisible by either 2 or 3

  $2, 3, 4, 6, 8, 9, 10, 12, 14, 15, \ldots$

$$a_n = \begin{cases} \lfloor n/4 \rfloor \cdot 6 + 2 & \text{if } n \bmod 4 = 1 \\ \lfloor n/4 \rfloor \cdot 6 + 3 & \text{if } n \bmod 4 = 2 \\ \lfloor n/4 \rfloor \cdot 6 + 4 & \text{if } n \bmod 4 = 3 \\ \lfloor n/4 \rfloor \cdot 6 + 6 & \text{if } n \bmod 4 = 0 \end{cases}$$

<u>Def.</u> If $n \div d = x$ rem r

then $x = \lfloor n/d \rfloor$

$r = n \bmod d$

$2, 3, 4, 6$

$\xrightarrow{+6}$ $8, 9, 10, 12$

$+6$

Try it out

- Find a simple formula for the following patterns
- 2, 5, 10, 17, 26, 37, 50, …

1, 4, 9, 16, 25, 36, 49, ...

$a_n = n^2 + 1$

$a_1 = 1 + 1 = 2$

$a_2 = 4 + 1 = 5$

$a_3 = 9 + 1 = 10$

$a_4 = 16 + 1 = 17$

A: $a_n = 10^{n-1}$
B: $a_n = 2n + 3$
C: $a_n = n^2 + 1$
D: $a_n = \sqrt{n} + 1$
E: None of the above

- 1, 10, 100, 1000, 10000, 100000, …

$a_n = 10\, a_{n-1}$

$a_1 = 1$ $a_3 = 100$

$a_2 = 10$ $a_4 = 1000$

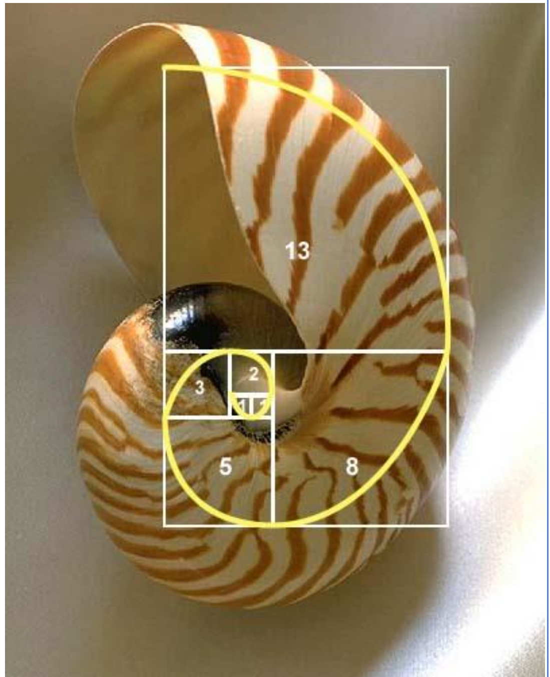
$a_n = 10^{n-1}$

- 1, 1, 2, 3, 5, 8, 13, 21, 34, …

Fibonacci sequence

$a_n = a_{n-1} + a_{n-2}$, $a_1 = 1$ $a_2 = 1$

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$



What about the primes?

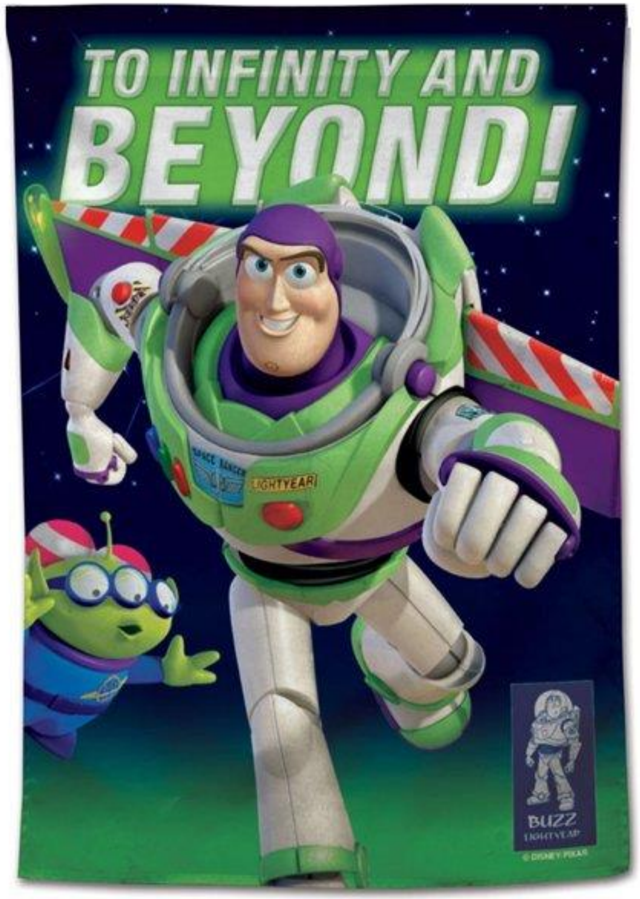
2	3	5	7	11	13	17	19	23	29
31	37	41	43	47	53	59	61	67	71
73	79	83	89	97	101	103	107	109	113
127	131	137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223	227	229
233	239	241	251	257	263	269	271	277	281
283	293	307	311	313	317	331	337	347	349
353	359	367	373	379	383	389	397	401	409
419	421	431	433	439	443	449	457	461	463
467	479	487	491	499	503	509	521	523	541
547	557	563	569	571	577	587	593	599	601
607	613	617	619	631	641	643	647	653	659
661	673	677	683	691	701	709	719	727	733
739	743	751	757	761	769	773	787	797	809
811	821	823	827	829	839	853	857	859	863
877	881	883	887	907	911	919	929	937	941
947	953	967	971	977	983	991	997	1009	1013
1019	1021	1031	1033	1039	1049	1051	1061	1063	1069

A few questions about primes

- How many prime numbers exist?
- If p is a prime, what is the next prime q, where $q > p$?
- Sometimes, when p is a prime, all of the numbers immediately following it are not. This is called a prime desert. How often does this happen?
- Sometimes, when p is a prime, so is $p + 2$. This is called a twin prime. How often does this happen?
- If I pick a random large number between n and m, what is the chance that it'll be a prime?

How many prime numbers are there?

A: Infinity!
B: 281213943920211239
C: A lot, but not infinity
D: Primes don't exist
E: None of the above



Direct proofs of infinity

- Ways of proving infinitely many items.
- Proof there are infinite natural numbers:
  - One way is to show that there is an infinite sequence without repetitions which is all natural numbers.

1, 2, 3, 4, 5, 6, 7, 8, ...

or can use 1, 3, 5, 7, 9, 11, ... which is also an infinite seq.

- Infinite even numbers:

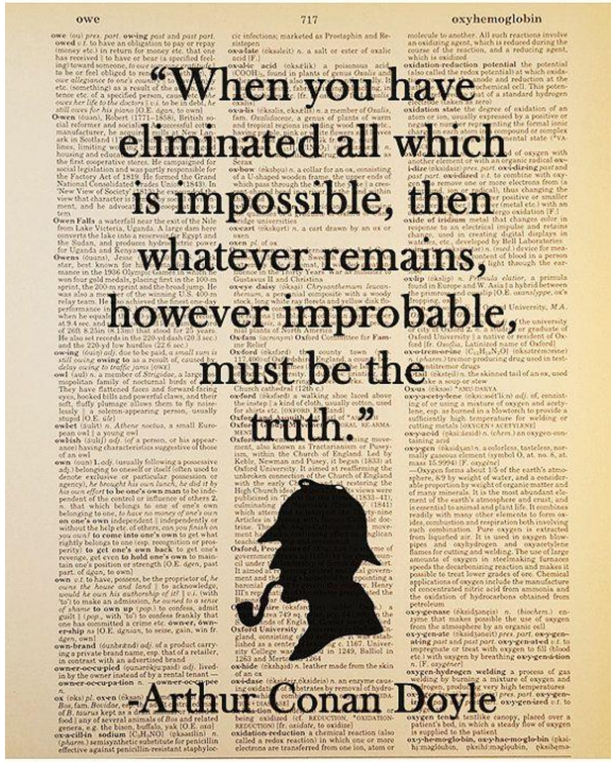
Ex. 2, 4, 6, 8, 10, 12, 14

Or. 2, 6, 10, 14, 18, ...

- This only works because we can figure out a simple formula for these sequences, but we don't know any formula even for a subset of the prime numbers.

Proof by contradiction

- Example:
  1. Mr. Body was murdered in the Billiards Room with a candlestick.
  2. Everyone but Colonel Mustard was in the Dining Room when the murder happened.
  3. Suppose Colonel Mustard was not the killer.
  - Then no one murdered Mr. Body.
  - Contradiction!
  - Therefore one of the facts must be wrong.
  - If we are sure of the first two facts, then the third one must be wrong.
  - If the third fact is wrong, then Colonel Mustard must be the killer.



Proof by contradiction for infinite evens

Suppose we have a finite set of even numbers.

Then must be largest even number $x = 2n$

e.g. $\{2, 4, 6, 8, \ldots, 100, \ldots, 2n\}$

But if $x = 2n$ is even, then $x+2 = 2n+2 = 2(n+1)$ is also even.

But then the largest even number is not x.

Contradiction! $\rightarrow\mid\leftarrow$

Therefore, the number of evens is infinite.

Proof of infinite primes

Suppose we have finitely many prime numbers.

Let's call them $p_1, \dots, p_n$.

We know every other number has a prime factorization, and therefore must be divisible by one of these primes.

Consider $x = p_1 \cdot p_2 \cdot p_3 \cdots p_n + 1$

But $x \div p_i$ has remainder 1 for any of these primes. So it is not divisible

Contradiction! $\rightarrow\leftarrow$

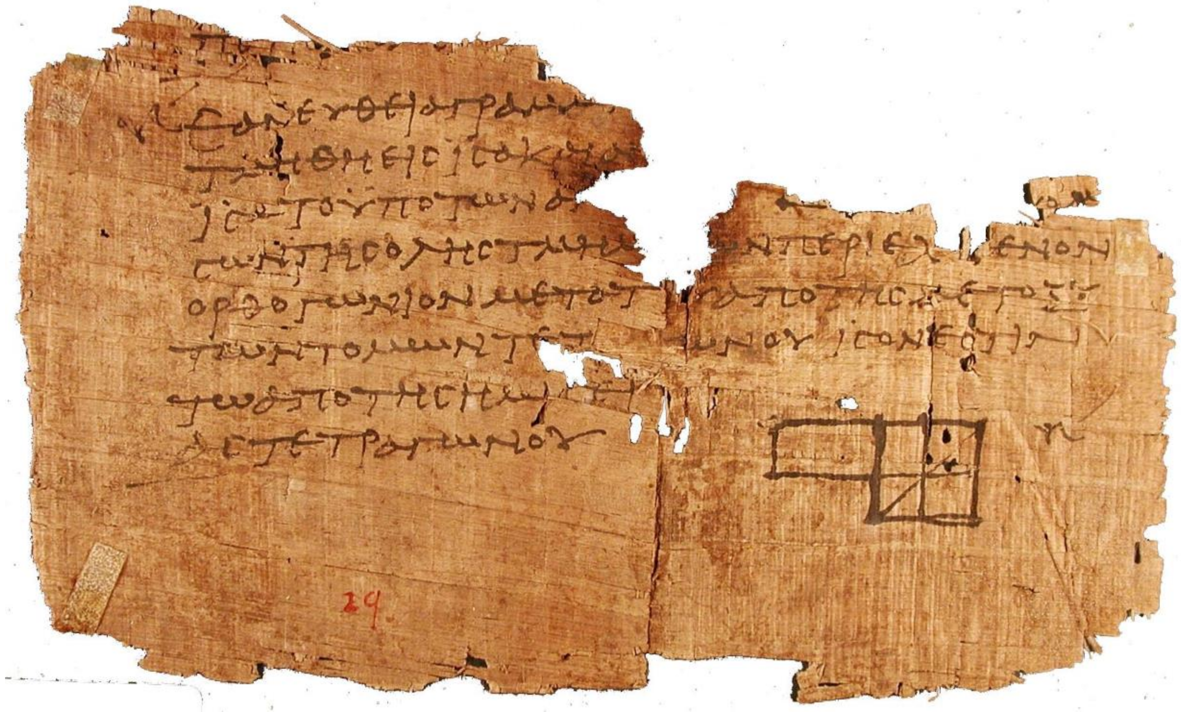
Therefore we have infinitely many primes.

History of operations

- Negative numbers were invented circa 202 BCE – 220 CE in China.
- Multiplication was invented around 4000 BCE by the Babylonians.
- Direct division was invented around 1500 BCE by the Egyptians.
- Modern long division was invented by Henry Briggs at Oxford, who lived from 1560-1630 CE.
- Sieve of Eratosthenes was known around 276 BCE – 194 BCE.

When did we prove infinitely many prime numbers exist?

A: Before 1000 BCE
B: 1000 BCE to 1000 CE
C: 1000 CE to 1500 CE
D: 1500 CE to 1800 CE
E: After 1800 CE



Euclid’s Elements, a textbook written by an ancient Greek mathematician around 300 BCE in Alexandria, Ptolemaic Egypt.

Prime deserts

- Claim: no matter how large a value of n, you can find a sequence of n numbers in a row such that none of them are prime.
- Factorial: $m! = 1 \times 2 \times 3 \times \cdots \times m$

Ex. $3! = 1 \times 2 \times 3 = 6$ $6! = 720$

- Sol: $(n+1)! + 2$ to $(n+1)! + (n+1)$
- Ex. For $n = 42$, consider $43!$

$$\left.\begin{array}{l} 43!+2 \text{ is divisible by } 2 \\ 43!+3 \text{ is divisible by } 3 \\ \vdots \\ 43!+43 \text{ is divisible by } 43 \end{array}\right\}$$

42 non-prime (composite) numbers in a row

Mango Sensation
The Dessert Kitchen,
73 Harbord St., Toronto

Other known facts about prime numbers

- Bertrand's postulate / Bertrand-Chebyshev theorem (proved 1852): For any integer $n > 3$, there always exists at least one prime number p between n and $2n$.

Ex. between 5 and 10, must be a prime (7)
between 1,000,000 and 2,000,000, must be a prime

- Prime Number Theorem [Hadamard, 1895; Poussin, 1896]: There are approximately $\frac{n}{\ln n}$ primes between 2 and n.

b/t 2 and 1,000,000, approx 72,382 primes
actually 78,498

b/t 2 and 2,000,000, approx 137,849 primes
actually 148,933

Do primes necessarily get further apart?

- Answer: No.
- Yitang Zhang proved in 2013 that there are infinitely many pairs of prime numbers that differ by 70 million or less.
- Current best proof is that there are infinitely many pairs of prime numbers that differ by 246 or less.
- Twin prime conjecture: there are infinitely many pairs of prime numbers that differ by exactly 2.

Professor Yitang Zhang, UNH (now UCSB)

Examples:

3,5

5,7

11,13

17,19

…

$$2996863034895 \cdot 2^{1290000} \pm 1$$