Using Factorizations
Lecture 4c: 2022-02-02

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Multiplication

- Prime factorizations are intrinsically tied to multiplication, and require just adding exponents.

$$33,880 = 2^3 \cdot 5 \cdot 7 \cdot 11^2$$

$$31,350 = 2 \cdot 3 \cdot 5^2 \cdot 11 \cdot 19$$

$$\Rightarrow 33,880 \times 31,350 = (2^3 \cdot 5 \cdot 7 \cdot 11^2)(2 \cdot 3 \cdot 5^2 \cdot 11 \cdot 19)$$

$$= \underbrace{2^4 \cdot 5^3 \cdot 7 \cdot 11^3 \cdot 19}_{\text{prime factorization}}$$

General rule for multiplication

- Let

$$n = 2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdot 7^{a_7} \dots$$
$$m = 2^{b_2} \cdot 3^{b_3} \cdot 5^{b_5} \cdot 7^{b_7} \dots$$

- Then

$$nm = 2^{a_2+b_2} \cdot 3^{a_3+b_3} \cdot 5^{a_5+b_5} \cdot 7^{a_7+b_7} \dots$$

Ex.

$$33{,}880 = \underline{2^3 \cdot 5 \cdot 7 \cdot 11^2} = 2^3 \cdot 3^0 \cdot 5^1 \cdot 7^1 \cdot 11^2 \cdot 13^0 \cdot 17^0 \cdot 19^0$$

$$31{,}350 = \underline{2 \cdot 3 \cdot 5^2 \cdot 11 \cdot 19} = 2^1 \cdot 3^1 \cdot 5^2 \cdot 7^0 \cdot 11^1 \cdot 13^0 \cdot 17^0 \cdot 19^1$$

$$\text{product} = 2^{1+3} \cdot 3^{0+1} \cdot 5^{1+2} \cdot 7^{1+0} \cdot 11^{2+1} \cdot 13^{0+0} \cdot 17^{0+0} \cdot 19^{0+1}$$

Try it out

$$120 = 2^3 \cdot 3 \cdot 5$$
$$132 = 2^2 \cdot 3 \cdot 11$$
$$189 = 3^3 \cdot 7$$

- What is the prime factorization of the following?
- $120 \cdot 132$ $= 2^3 \cdot 3 \cdot 5 \cdot 2^2 \cdot 3 \cdot 11 = 2^{2+3} \cdot 3^{1+1} \cdot 5 \cdot 11$
  $= 2^5 \cdot 3^2 \cdot 5 \cdot 11$
- $132 \cdot 189$ $= (2^2 \cdot 3 \cdot 11)(3^3 \cdot 7)$
  $= 2^2 \cdot 3^4 \cdot 7 \cdot 11$. A
- $120 \cdot 189$

$= 2^3 \cdot 3^4 \cdot 5 \cdot 7$. C

$2^3 \, 3^{1+3} \cdot 5 \cdot 7$

$2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 3 \cdot 3 \cdot 3 \cdot 7$

A: $2^2 \cdot 3^4 \cdot 7 \cdot 11$
B: $2^5 \cdot 3^2 \cdot 5 \cdot 11$
C: $2^3 \cdot 3^4 \cdot 5 \cdot 7$
D: $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11$
E: None of the above

Division via Factorization

- Division is opposite of multiplication.
- Can use power rules to show division of prime factorizations is the same as subtracting exponents.
- Let

$$n = 2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdot 7^{a_7} \dots$$
$$m = 2^{b_2} \cdot 3^{b_3} \cdot 5^{b_5} \cdot 7^{b_7} \dots$$

- Then

$$\frac{n}{m} = 2^{a_2 - b_2} \cdot 3^{a_3 - b_3} \cdot 5^{a_5 - b_5} \cdot 7^{a_7 - b_7} \dots$$

- And n is divisible by m precisely when $a_2 \geq b_2, a_3 \geq b_3, \dots$

Example

$$\frac{33{,}880}{31{,}350} = \frac{2^3 \cdot 5 \cdot 7 \cdot 11^2}{2 \cdot 3 \cdot 5^2 \cdot 11 \cdot 19} = \frac{2^2 \cdot 7 \cdot 11}{3 \cdot 5 \cdot 19}$$

$$= 2^{3-1}\, 3^{-1}\, 5^{1-2} \cdot 7 \cdot 11^{2-1} \cdot 19^{-1}$$

$$= 2^2\, 3^{-1}\, 5^{-1}\, 7 \cdot 11 \cdot 19^{-1}$$

$$\frac{120}{60} = \frac{2^3 \cdot 3 \cdot 5}{2^2 \cdot 3 \cdot 5} = 2^{3-1}\, 3^{1-1}\, 5^{1-1} = 2 \cdot 3^0 \cdot 5^0 = 2.$$

Try it out

- For the following pairs of numbers, say if n is divisible by m.

A: Yes
B: No
E: None of the above

- $n = 2^3 \cdot 3 \cdot 7^2 \cdot 13,\ m = 2^2 \cdot 3 \cdot 7$

$$\frac{n}{m} = \frac{2^3 \cdot 3 \cdot 7^2 \cdot 13}{2^2 \cdot 3 \cdot 7} = 2 \cdot 7 \cdot 13$$

A. Yes

- $n = 2^4 \cdot 11 \cdot 19,\ m = 2^2 \cdot 3 \cdot 11$

$$\frac{n}{m} = \frac{2^4 \cdot 11 \cdot 19}{2^3 \cdot 3 \cdot 11} = \frac{2 \cdot 19}{3}$$

No

$$\frac{12}{9} = \frac{2 \cdot 2 \cdot \cancel{3}}{3 \cdot \cancel{3}} = \frac{4}{3}$$

- $n = 2^2 \cdot 3 \cdot 7^2 \cdot 13,\ m = 2^3 \cdot 3 \cdot 7$

3 > 2 in powers of 2.

No.

Counting divisors

- How many numbers divide 24?

$24 \div 1 = 24$ ✓
$24 \div 2 = 12$ ✓
$24 \div 3 = 8$ ✓
$24 \div 4 = 6$ ✓
$24 \div 5 = 4 \text{ r } 4$
$24 \div 6 = 4$ ✓
$24 \div 7 = 3 \text{ r } 3$
$24 \div 8 = 3$ ✓

$24 \div 9 = 2 \text{ r } 6$
$24 \div 10 = 2 \text{ r } 4$
$24 \div 11 = 2 \text{ r } 2$
$24 \div 12 = 2$ ✓
$24 \div 13 = 1 \text{ r } 11$
$24 \div 14 = 1 \text{ r } 10$
$\vdots$
$24 \div 24 = 1$ ✓

A: 4
B: 6
C: 8
D: 10
E: None of the above

- Counting up is slow and time-consuming.

Counting divisors using factorization

- How many divisors does 24 have?

$$24 = 2 \cdot 12$$
$$= 2 \cdot 2 \cdot 6$$
$$= 2 \cdot 2 \cdot 2 \cdot 3$$
$$= 2^3 \cdot 3$$

$2^0 \cdot 3^0 = 1$ $\quad$ $2^0 \cdot 3^1 = 3$

$2^1 \cdot 3^0 = 2$ $\quad$ $2^1 \cdot 3^1 = 6$

$2^2 \cdot 3^0 = 4$ $\quad$ $2^2 \cdot 3^1 = 12$

$2^3 \cdot 3^0 = 8$ $\quad$ $2^3 \cdot 3^1 = 24$

All divisors must be of the form $2^a 3^b$,

$$a \le 3$$
$$b \le 1$$

$\Rightarrow a \in \{0, 1, 2, 3\} \leftarrow 4$

$b \in \{0, 1\} \longleftarrow 2$

$4 \times 2 = 8$

divisors

Another example

- How many numbers divide 756?
- Step 1: factor 756

$$756 = 2 \cdot 378 = 2^2 \cdot 189 = 2^2 \cdot 3 \cdot 63 = 2^2 \cdot 3^2 \cdot 21$$
$$= 2^2 \cdot 3^3 \cdot 7^1$$

- Step 2: multiply together 1+powers

$$(2+1)(3+1)(1+1) = 3 \cdot 4 \cdot 2 = 24 \text{ divisors}$$

Ex $2^1 \cdot 3^2 \cdot 7 = 18 \cdot 7 = 70+56 = 126$ is a divisor

Try it out

- How many divisors does 384 have?

A: 4
B: 8
C: 12
D: 16
E: None of the above

$384 = 2 \cdot 192 = \underbrace{2 \cdot 2}_{2^2} \cdot 96$

$= 2^2 \cdot 96 = 2^2 \cdot 2 \cdot 48$

$= 2^3 \cdot 48 = 2^3 \cdot 2 \cdot 24$

$= 2^4 \cdot 24$

$= 2^5 \cdot 12$

$= 2^6 \cdot 6$

$= \underbrace{2^7 \cdot 3^1}$

$(7+1)(1+1) = \underline{8 \cdot 2} = 16$

Say d is a divisor:
then $d = 2^a 3^b$, $a \le 7$, $b \le 1$

8 numbers $\{0, 1, 2, 3, 4, 5, 6, 7\} \le 7$

2 numbers $\{0, 1\} \le 1$

gcd via factorization

- Suppose d divides n. Then prime factorization of d must have exponents less than or equal to prime factorization of n.
- If d divides both n and m, then all of the exponents must be less than or equal to those in both n and m.
- The greatest common divisor thus has exponents exactly equal to the smaller of the exponents in n and m.

$24 = 2^3 \cdot 3^1$

$54 = 2^1 \cdot 3^3$

$\gcd(24, 54) = 2^1 \cdot 3^1 = 6$

$54 = 24 \cdot 2 + 6$

$24 = 6 \cdot 4$

$\gcd(24, 6) = 6$

$21 = 3 \cdot 7 = 2^0 \cdot 3^1 \cdot 7^1$

$14 = 2 \cdot 7 = 2^1 \cdot 3^0 \cdot 7^1$

$\gcd(21, 14) = 2^0 \cdot 3^0 \cdot 7^1 = 7$

lcm via factorization

- Suppose m is a multiple of a. Then prime factorization of m must have exponents greater than or equal to prime factorization of a.
- If m is a multiple of both a and b, then all of the exponents must be greater than or equal to those in both a and b.
- The least common multiple thus has exponents exactly equal to the larger of the exponents in n and m.

$24 = 2^3 \cdot 3$

$54 = 2 \cdot 3^3$

$\text{lcm} = 2^3 \cdot 3^3 = 8 \cdot 27 = 216$

24 48 72 96 120 144 168 192 216

54 108 162 216

Showing $\operatorname{lcm}(a, b) \cdot \gcd(a, b) = ab$

- The lcm gets all the smaller exponents, and the gcd gets all the larger exponents, so multiplied together, you get all of the original exponents in a and b.