Using Factorizations Lecture 4c: 2022-02-02

MAT A02 – Winter 2022 – UTSC

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Multiplication

- Prime factorizations are intrinsically tied to multiplication, and require just adding exponents.

General rule for multiplication

- Let

$$n = 2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdot 7^{a_7} \dots$$
$$m = 2^{b_2} \cdot 3^{b_3} \cdot 5^{b_5} \cdot 7^{b_7} \dots$$

- Then

$$nm = 2^{a_2+b_2} \cdot 3^{a_3+b_3} \cdot 5^{a_5+b_5} \cdot 7^{a_7+b_7} \dots$$

Try it out

$$120 = 2^3 \cdot 3 \cdot 5$$
$$132 = 2^2 \cdot 3 \cdot 11$$
$$189 = 3^3 \cdot 7$$

- What is the prime factorization of the following?
- $120 \cdot 132$
- $132 \cdot 189$
- $120 \cdot 189$

A: $2^2 \cdot 3^4 \cdot 7 \cdot 11$
B: $2^5 \cdot 3^2 \cdot 5 \cdot 11$
C: $2^3 \cdot 3^4 \cdot 5 \cdot 7$
D: $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11$
E: None of the above

Division via Factorization

- Division is opposite of multiplication.
- Can use power rules to show division of prime factorizations is the same as subtracting exponents.
- Let

$$n = 2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdot 7^{a_7} \ldots$$
$$m = 2^{b_2} \cdot 3^{b_3} \cdot 5^{b_5} \cdot 7^{b_7} \ldots$$

- Then

$$\frac{n}{m} = 2^{a_2 - b_2} \cdot 3^{a_3 - b_3} \cdot 5^{a_5 - b_5} \cdot 7^{a_7 - b_7} \ldots$$

- And n is divisible by m precisely when $a_2 \geq b_2, a_3 \geq b_3, \ldots$

Example

Try it out

- For the following pairs of numbers, say if n is divisible by m.
- $n = 2^3 \cdot 3 \cdot 7^2 \cdot 13,\ m = 2^2 \cdot 3 \cdot 7$

A: Yes
B: No
E: None of the above

- $n = 2^4 \cdot 11 \cdot 19,\ m = 2^2 \cdot 3 \cdot 11$

- $n = 2^2 \cdot 3 \cdot 7^2 \cdot 13,\ m = 2^3 \cdot 3 \cdot 7$

Counting divisors

- How many numbers divide 24?

A: 4
B: 6
C: 8
D: 10
E: None of the above

- Counting up is slow and time-consuming.

Counting divisors using factorization

- How many divisors does 24 have?

Another example

- How many numbers divide 756?
- Step 1: factor 756
- Step 2: multiply together 1+powers

Try it out

- How many divisors does 384 have?

A: 4
B: 8
C: 12
D: 16
E: None of the above

gcd via factorization

- Suppose d divides n. Then prime factorization of d must have exponents less than or equal to prime factorization of n.
- If d divides both n and m, then all of the exponents must be less than or equal to those in both n and m.
- The greatest common divisor thus has exponents exactly equal to the smaller of the exponents in n and m.

lcm via factorization

- Suppose m is a multiple of a. Then prime factorization of m must have exponents greater than or equal to prime factorization of a.
- If m is a multiple of both a and b, then all of the exponents must be greater than or equal to those in both a and b.
- The least common multiple thus has exponents exactly equal to the larger of the exponents in n and m.

Showing $\operatorname{lcm}(a,b)\cdot\gcd(a,b)=ab$

- The lcm gets all the smaller exponents, and the gcd gets all the larger exponents, so multiplied together, you get all of the original exponents in a and b.

Try it out

- What is the greatest common divisor of 3072 and 896?
- Step 1: what is the prime factorization of 3072?

A: $2^8 \cdot 5$
B: $2^7 \cdot 7$
C: $2^{10} \cdot 3$
D: $2^5 \cdot 3^5$
E: None of the above

- Step 2: what is the prime factorization of 896?
- Step 3: get the smallest of each exponent.

A: $2^5 = 32$
B: $2^7 = 128$
C: $2^7 \cdot 3 \cdot 5 = 1920$
D: $2^{10} \cdot 3 \cdot 7 = 21504$
E: None of the above

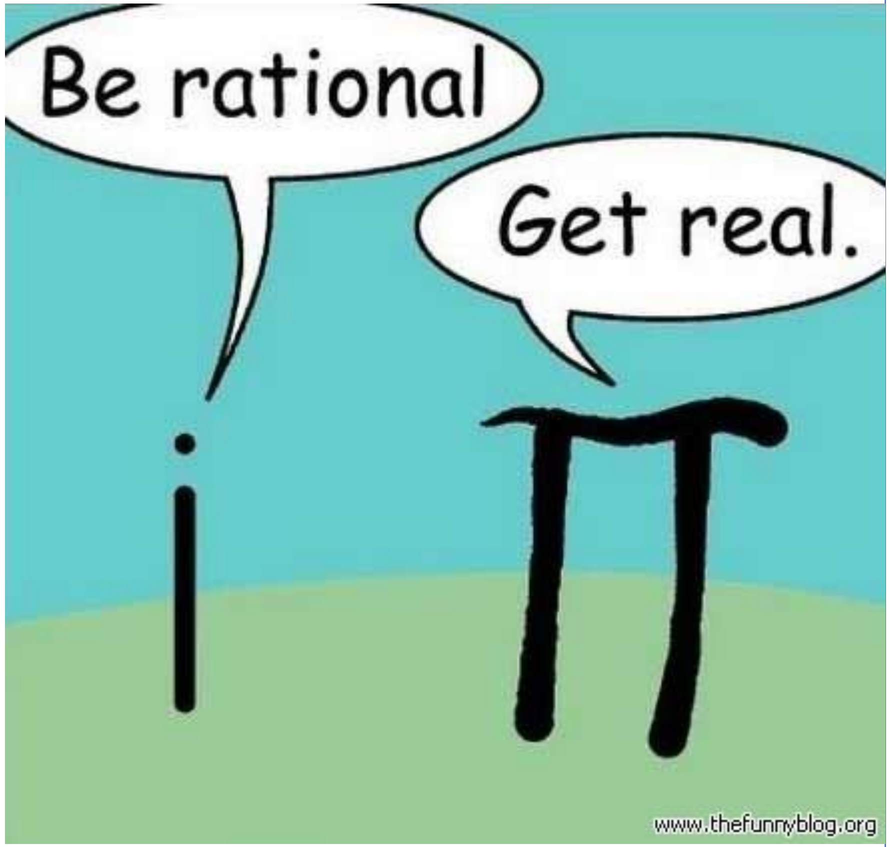
Try it out

- How many positive integers divide both 3072 and 896?
  Hint: if an integer divides both, then it must divide their gcd.

A: 4
B: 8
C: 12
D: 16
E: None of the above

Rational numbers

- Any number that can expressed as a fraction where the numerator and denominator are both whole numbers is called "rational".
- Otherwise the number is called "irrational".
- Are there any irrational numbers?

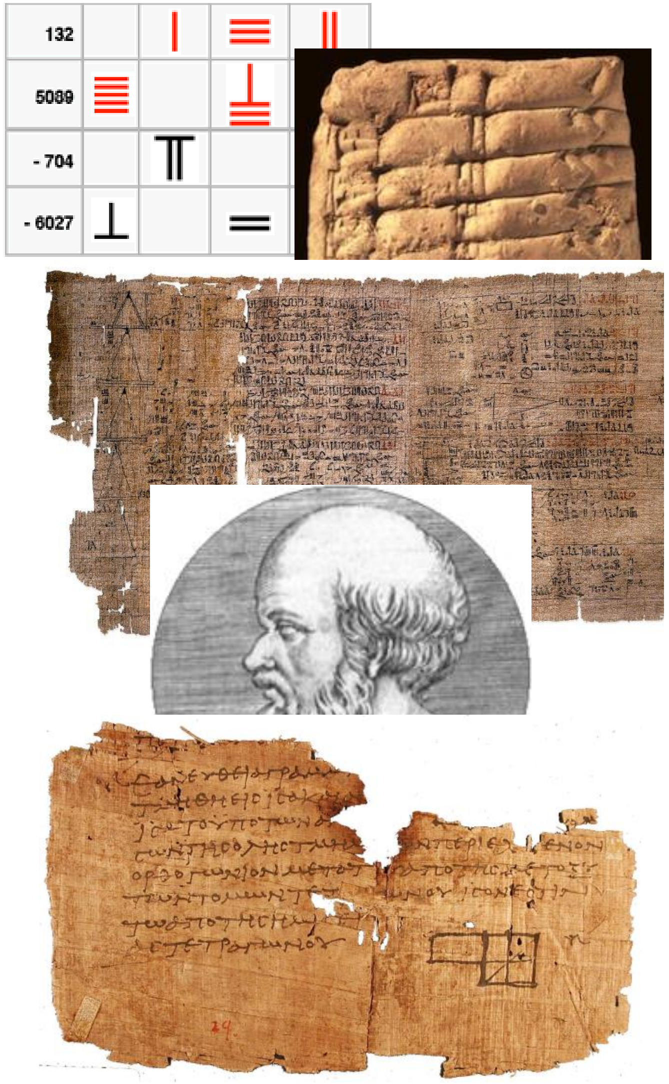


$\sqrt{2}$ is irrational

- Claim: there is no fraction $\frac{n}{m}$ such that $\left(\frac{n}{m}\right)^2 = 2$. ($n, m$ integers)

Math history

- Negative numbers were invented circa 202 BCE – 220 CE in China.
- Multiplication was invented around 4000 BCE by the Babylonians.
- Direct division was invented around 1500 BCE by the Egyptians.
- Sieve of Eratosthenes was known around 276 BCE – 194 BCE.
- Infinitude of primes was written down in Euclid’s Elements from 300 BCE in Alexandria, Egypt.



History of irrational numbers

- When did we prove that $\sqrt{2}$ is irrational, that it could not be written as a fraction of whole numbers?

A: Before 1000 BCE
B: 1000 BCE to 1000 CE
C: 1000 CE to 1500 CE
D: 1500 CE to 1800 CE
E: After 1800 CE

Hippasus of Metapontum, 530-450 BCE
One of the Pythagoreans, sometimes credited with the discovery of irrational numbers.
(the Pythagoreans knew of irrational numbers, but who exactly proved it is a bit unclear)