LCM and GCD via factorization
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Greatest common divisor

- Suppose d divides n. Then prime factorization of d must have exponents less than or equal to prime factorization of n.
- If d divides both n and m, then all of the exponents must be less than or equal to those in both n and m.
- The greatest common divisor thus has exponents exactly equal to the smaller of the exponents in n and m.

$54 = 24 \cdot 2 + 6$

$24 = 6 \cdot 4$

$\gcd(24, 54) = 6$

$24 = 2^3 \cdot 3$

$54 = 2 \cdot 3^3$

$\gcd(24, 54) = 2^1 \cdot 3^1 = 6$

Least common multiple

- Suppose m is a multiple of a. Then prime factorization of m must have exponents greater than or equal to prime factorization of a.
- If m is a multiple of both a and b, then all of the exponents must be greater than or equal to those in both a and b.
- The least common multiple thus has exponents exactly equal to the larger of the exponents in n and m.

$24 = 2^3 \cdot 3$

$54 = 2 \cdot 3^3$

$\text{lcm}(24, 54) = 2^3 \cdot 3^3$

$= 8 \cdot 27 = 216$

24, 48, 72, 96, 120, 144, 168, 192, 216

54, 108, 162, 216

$$\text{lcm}(a, b) \cdot \gcd(a, b) = ab$$

- The lcm gets all the smaller exponents, and the gcd gets all the larger exponents, so multiplied together, you get all of the original exponents in a and b.

$$a = 2^{c_2} 3^{c_3} 5^{c_5} 7^{c_7} 11^{c_{11}} \cdots$$

$$b = 2^{d_2} 3^{d_3} 5^{d_5} 7^{d_7} 11^{d_{11}} \cdots$$

$$ab = 2^{c_2+d_2} 3^{c_3+d_3} 5^{c_5+d_5} \cdots$$

Note $x + y = \min(x, y) + \max(x, y)$

$$= \left(2^{\min(c_2, d_2)} 3^{\min(c_2, d_2)} \cdots\right)\left(2^{\max(c_2, d_2)} 3^{\max(c_2, d_2)} \cdots\right)$$

$$= \gcd(a, b) \cdot \text{lcm}(a, b)$$

Try it out

- What is the greatest common divisor of 3072 and 896?
- Step 1: what is the prime factorization of 3072?

$3072 = 1024 \cdot 3 = 2^{10} \cdot 3$

A: $2^8 \cdot 5$
B: $2^7 \cdot 7$
C: $2^{10} \cdot 3$
D: $2^5 \cdot 3^5$
E: None of the above

- Step 2: what is the prime factorization of 896?

$896 = 7 \cdot 128 = 2^7 \cdot 7$

A: $2^5 = 32$
B: $2^7 = 128$
C: $2^7 \cdot 3 \cdot 5 = 1920$
D: $2^{10} \cdot 3 \cdot 7 = 21504$
E: None of the above

- Step 3: get the smallest of each exponent.

$\gcd(896, 3072) = 2^7 \cdot \cancel{3^0} \cdot \cancel{7^0} = 128$