Irrational and imaginary numbers Lecture 5c: 2022-02-09

MAT A02 – Winter 2022 – UTSC

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What do mathematicians do?

- Invent new ways of counting/measuring things
  - Addition $5+3=8$
  - Multiplication $5\times 3=15$
  - Exponentiation $5^3=125$ $5^2=25$ $(-5)^2=25$
- Figure out how to reverse operations
  - Subtraction $8-3=5$
  - Division $15\div 3=5$
  - Roots $\sqrt[3]{125}=5$ $\sqrt{25}=\pm 5$

Careful, not always unique

- Invent new numbers to allow reversing stuff
  - Negative numbers -3
  - Fractions $\frac{1}{3}$
  - ???

$\sqrt{3}$

Do we need a number type for reversing exponentiation?

A: Yes

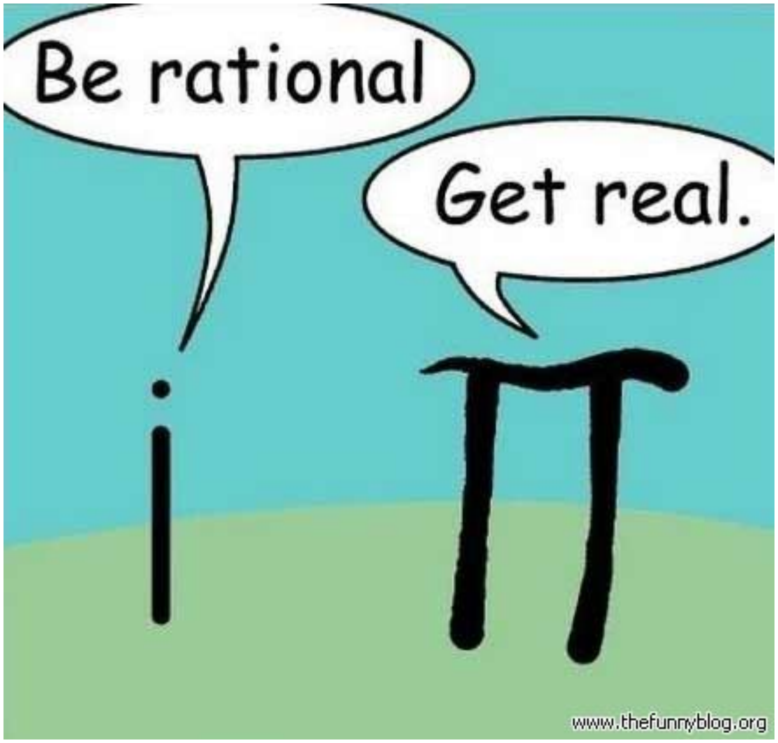
B: No

C: Yes, in fact more than 1

E: None of the above

Rational numbers ≈ fractions

- Any number that can expressed as a fraction where the numerator and denominator are both whole numbers is called “rational”.
- Otherwise the number is called “irrational”.



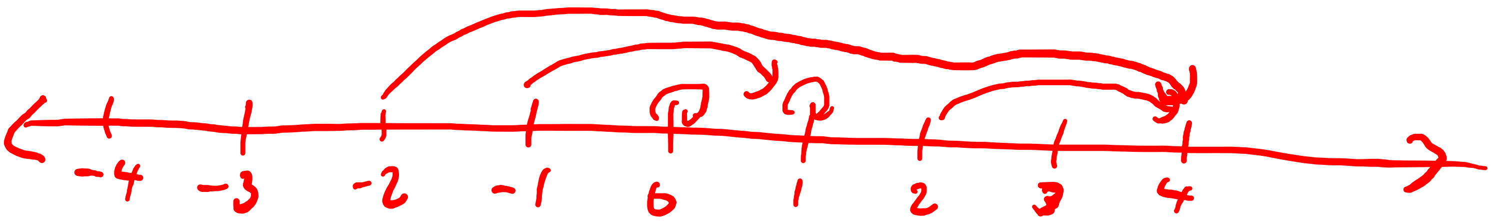
Ex. 20 is rational because $\frac{40}{2}$ is a fraction

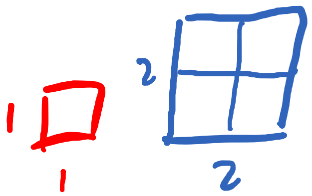
(so is $\frac{20}{1}$)

What about π? or $\sqrt{2}$?

What does it mean to “square”

$x^2 = x \cdot x$

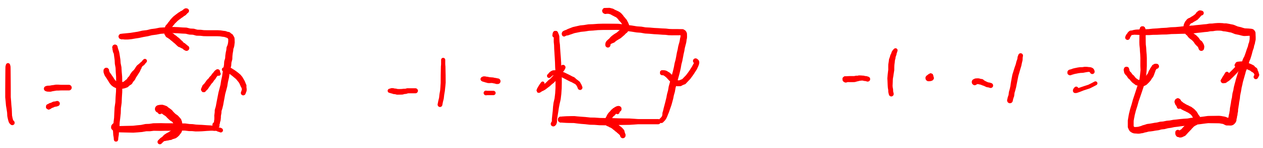




$2^2 = 4 \cdot 1^2$

A square with side-length 2 is 4 side-length 1 squares

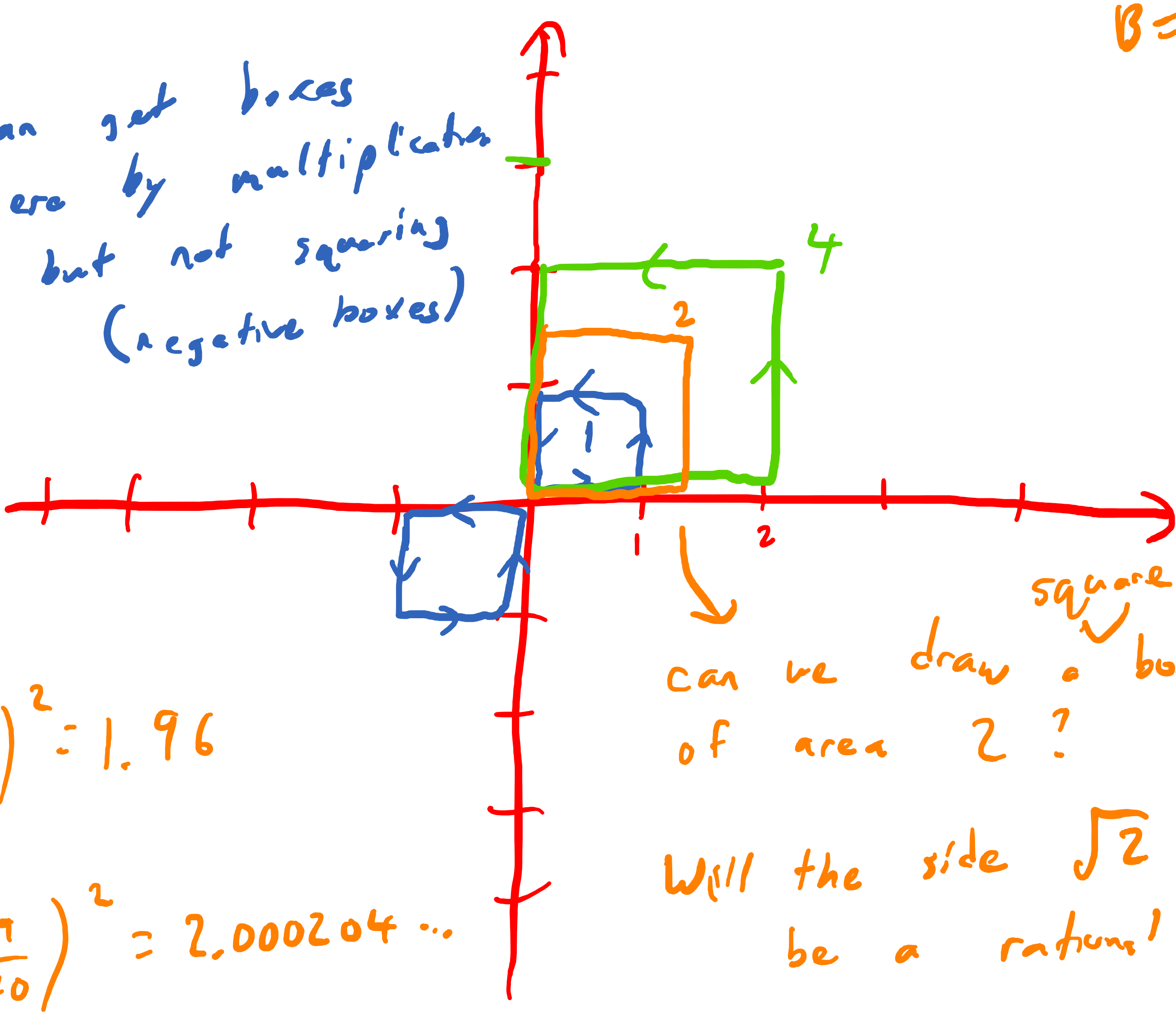
Negative numbers are like flipping



2D grid and squaring

A = Yes
B = No

Can get boxes here by multiplication but not squaring (negative boxes)



can we draw a square box of area 2 ?

Will the side $\sqrt{2}$ be a rational number?

$\left(\frac{7}{5}\right)^2 = 1.96$

$\left(\frac{99}{70}\right)^2 = 2.000204\ldots$

$\sqrt{2}$ is irrational

$n = 6 = 2^1 3^1$, a_2, a_3

- Claim: there is no fraction $\frac{n}{m}$ such that $\left(\frac{n}{m}\right)^2 = 2$. ($n$, m integers)

proof. Suppose there is $\left(\frac{n}{m}\right)^2 = 2$, n, m both integers.

Then $n^2 = 2m^2$

Use prime factorizations:

$n = 2^{a_2} 3^{a_3} \cdots$ $\quad$ $m = 2^{b_2} 3^{b_3} \cdots$

$\Rightarrow n^2 = 2^{2a_2} 3^{2a_3} \cdots$ $\quad$ $m^2 = 2^{2b_2} 3^{2b_3} \cdots$

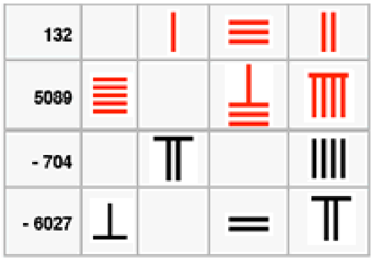
But $n^2 = 2 \cdot m^2 = 2^{2b_2+1} 3^{b_3} \cdots$

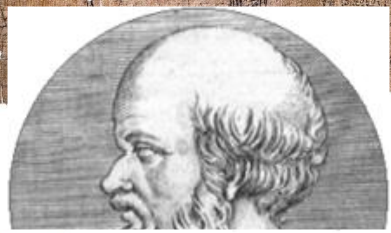
$\Rightarrow \underbrace{2a_2}_{\text{even}} = \underbrace{2b_2+1}_{\text{odd}}$, even $\neq$ odd, so contradiction!

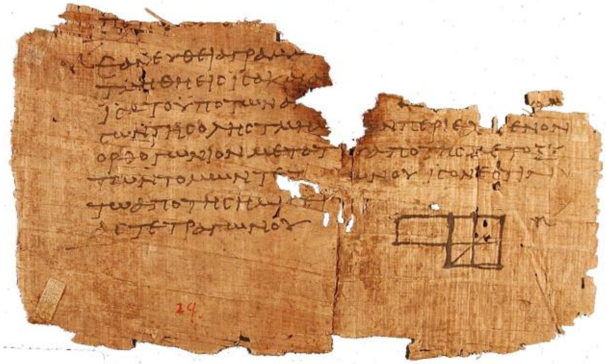
$\Rightarrow \sqrt{2}$ is irrational

Math history

- Negative numbers were invented circa 202 BCE – 220 CE in China.
- Multiplication was invented around 4000 BCE by the Babylonians.
- Direct division was invented around 1500 BCE by the Egyptians.
- Sieve of Eratosthenes was known around 276 BCE – 194 BCE.
- Infinitude of primes was written down in Euclid’s Elements from 300 BCE in Alexandria, Egypt.







History of irrational numbers

- When did we prove that $\sqrt{2}$ is irrational, that it could not be written as a fraction of whole numbers?

A: Before 1000 BCE
B: 1000 BCE to 1000 CE
C: 1000 CE to 1500 CE
D: 1500 CE to 1800 CE
E: After 1800 CE

Hippasus of Metapontum, 530-450 BCE
One of the Pythagoreans, sometimes credited with the discovery of irrational numbers.
(the Pythagoreans knew of irrational numbers, but who exactly proved it is a bit unclear)

Origins of a few irrational numbers

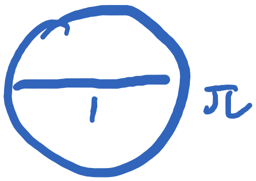
- Many roots of numbers:
  - Every square root $\sqrt{n}$, where n is not a perfect square, i.e. $n \neq m^2$, for a natural number m.

$\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8},$

$\sqrt[3]{2}, \sqrt[3]{3}, \sqrt[3]{4}, \ldots$

- Other parts of geometry, such as the ratio of a circle's circumference to its diameter:

$\pi \approx 3.14159265358979323846264338327950 \ldots$



- Some come from calculus and limits: Euler's number

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \approx 2.71828 \ldots$$

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

When was π proved irrational?

A: Before 1000 BCE
B: 1000 BCE to 1000 CE
C: 1000 CE to 1500 CE
D: 1500 CE to 1800 CE
E: After 1800 CE

Johann Heinrich Lambert, Swiss-French mathematician proved in the 1760s that π is irrational.

Legislating Pi by “squaring the circle”

- Which place tried (unsuccessfully) to pass a bill that effectively set $\pi = 3.2$?

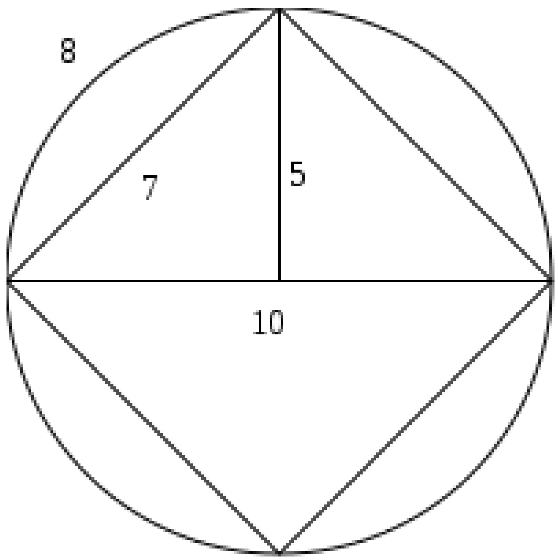
A: Indiana

B: Ontario

C: Kentucky

D: Texas

“Indiana Pi bill”



Physician Edward Goodwin’s model

1897 political cartoon

Passed the Indiana House of Representatives on February 6, 1897.

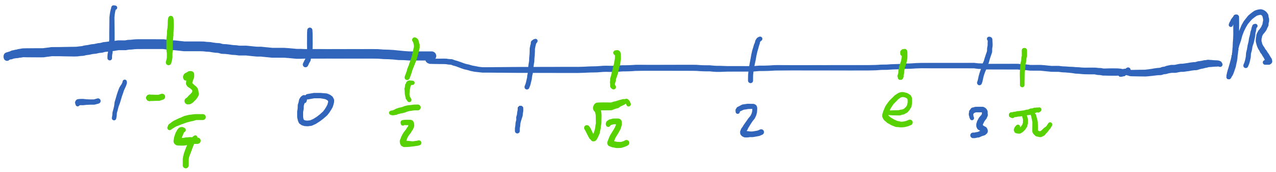
Ridiculed in the Indiana Senate after Purdue Professor Clarence Waldo explained why you can’t legislate math.

CLARENCE ABIATHAR WALDO,
A. M., Ph. D.,
Head Professor of Mathematics.

Real numbers

$\mathbb{R}$

- Any number that can be written as an infinite decimal on the number line is real.



- In order to fully understand and invent real numbers, we would need a better understanding of infinity, limits, and calculus.
- Is $0.9\overline{999}\cdots = 1$?

$1.0\overline{000}$

A: Yes
B: No
C: Depends on what infinity is
D: The question is ill-defined
E: None of the above

The same number can have multiple labels: $\frac{1}{2} = \frac{2}{4}$

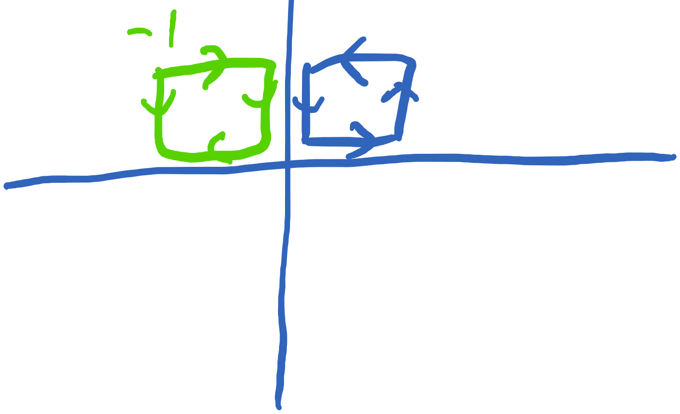
$\frac{1}{3} = 0.\overline{33}$ $\qquad \frac{3}{3} = 0.\overline{99} = 1$

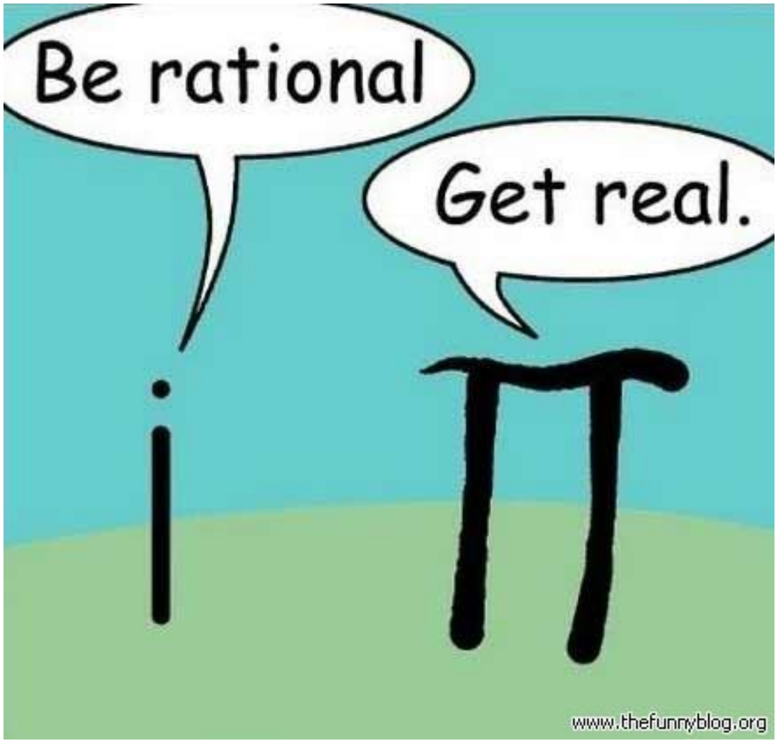
$$\begin{array}{r} 1.000000\cdots \\ -\,0.999999\cdots \\ \hline 0 = 0.\underbrace{0 \cdots 0 \cdots}_{\text{infinite}}01 \end{array}$$

Imaginary numbers

- What about square roots of negative numbers?

Recall:

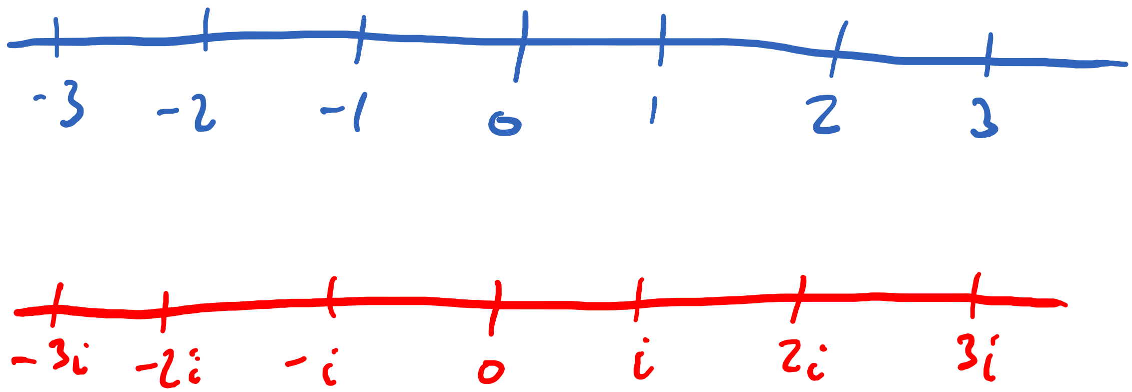




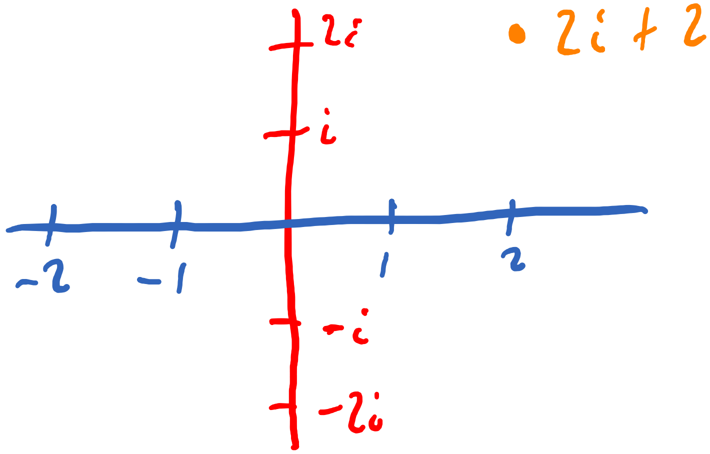
- Any positive number squared is positive.
- Any negative number squared is positive.
- Thus, square roots of negative numbers can't be either positive or negative, or on the real line at all.

New basic imaginary unit $i = \sqrt{-1}$

- Not on the real line, but we can imagine it as another imaginary line.



- And since it’s a different direction, we often choose to represent it as perpendicular in the complex plane.



Complex numbers

- Complex numbers $a + bi$ are a combination of a real number and an imaginary number.

$5 + 4i$

- Complex numbers are "complete" under taking square roots, addition, multiplication, etc.

$(1+2i) + (2-i) = 3 + i$

$(1+2i)(2-i) = 2 \underbrace{- i + 4i}_{} - 2i^2$

$= 2 + 3i + 2 = 4 + 3i$

$\sqrt{i} = \pm \frac{1}{\sqrt{2}}(1+i)$

Invention of complex numbers

A: Before 1000 BCE
B: 1000 BCE to 1000 CE
C: 1000 CE to 1500 CE
D: 1500 CE to 1800 CE
E: After 1800 CE

HIERONYMI CAR
DANI, PRÆSTANTISSIMI MATHE
MATICI, PHILOSOPHI, AC MEDICI,
ARTIS MAGNÆ,
SIVE DE REGVLIS ALGEBRAICIS,
Lib. unus. Qui & totius operis de Arithmetica, quod
OPVS PERFECTVM
inscripsit, est in ordine Decimus.

Ars Magna, 1545, by Gerolamo Cardano

- In the 15th and 16th centuries, mathematicians would have duels, challenging each other to solve problems publicly, like what is a solution to $5x^3 + 2x = 3$?
- Even when the final answer is real, sometimes you need to use imaginary numbers to find the answer.
- In modern day math, complex numbers are important whenever trigonometry appears, such as in electrical engineering.