Irrational and imaginary numbers Lecture 5c: 2022-02-09

MAT A02 – Winter 2022 – UTSC

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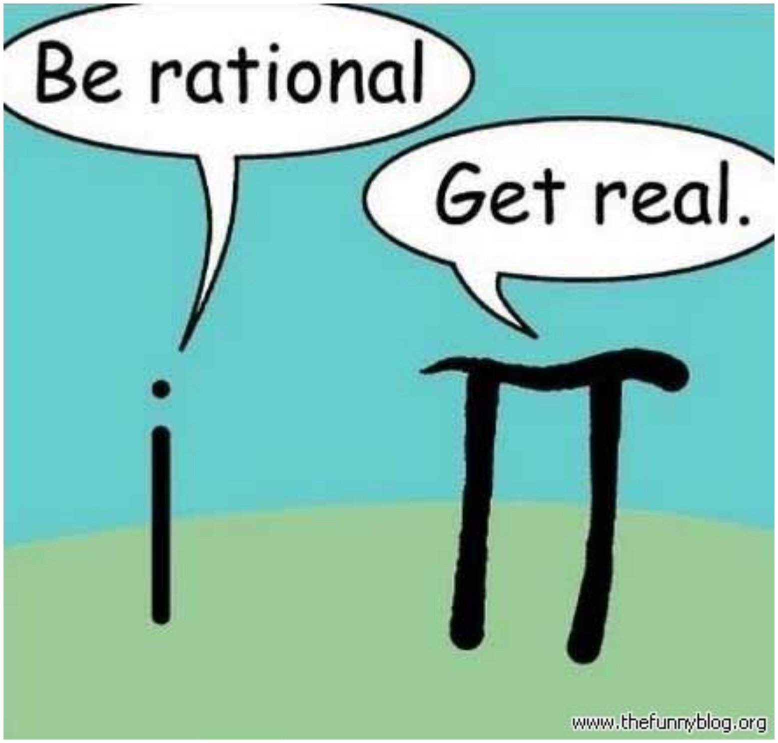
What do mathematicians do?

- Invent new ways of counting/measuring things
  - Addition
  - Multiplication
  - Exponentiation
- Figure out how to reverse operations
  - Subtraction
  - Division
  - Roots
- Invent new numbers to allow reversing stuff
  - Negative numbers
  - Fractions
  - ???

Do we need a number type for reversing exponentiation?
A: Yes
B: No
C: Yes, in fact more than 1
E: None of the above

Rational numbers ≈ fractions

- Any number that can expressed as a fraction where the numerator and denominator are both whole numbers is called “rational”.
- Otherwise the number is called “irrational”.



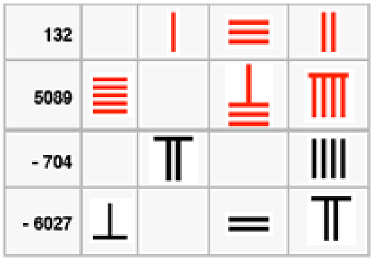
What does it mean to “square”

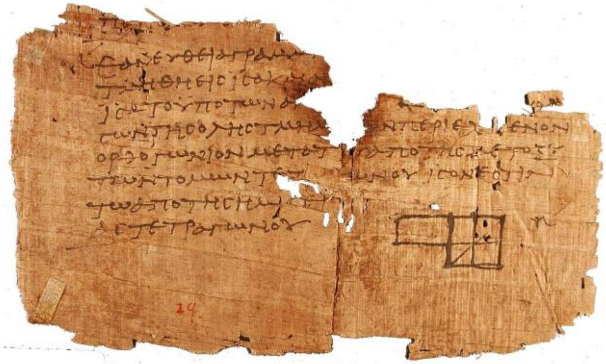
2D grid and squaring

$\sqrt{2}$ is irrational

- Claim: there is no fraction $\frac{n}{m}$ such that $\left(\frac{n}{m}\right)^2 = 2$. ($n$, m integers)

Math history





- Negative numbers were invented circa 202 BCE – 220 CE in China.
- Multiplication was invented around 4000 BCE by the Babylonians.
- Direct division was invented around 1500 BCE by the Egyptians.
- Sieve of Eratosthenes was known around 276 BCE – 194 BCE.
- Infinitude of primes was written down in Euclid’s Elements from 300 BCE in Alexandria, Egypt.

History of irrational numbers

- When did we prove that $\sqrt{2}$ is irrational, that it could not be written as a fraction of whole numbers?

A: Before 1000 BCE
B: 1000 BCE to 1000 CE
C: 1000 CE to 1500 CE
D: 1500 CE to 1800 CE
E: After 1800 CE

Origins of a few irrational numbers

- Many roots of numbers:
  - Every square root $\sqrt{n}$, where n is not a perfect square, i.e. $n \neq m^2$, for a natural number m.

- Other parts of geometry, such as the ratio of a circle's circumference to its diameter:
  $\pi \approx 3.14159265358979323846264338327950 \ldots$

- Some come from calculus and limits: Euler's number
$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \approx 2.71828 \ldots$$

When was π proved irrational?

A: Before 1000 BCE
B: 1000 BCE to 1000 CE
C: 1000 CE to 1500 CE
D: 1500 CE to 1800 CE
E: After 1800 CE

Real numbers

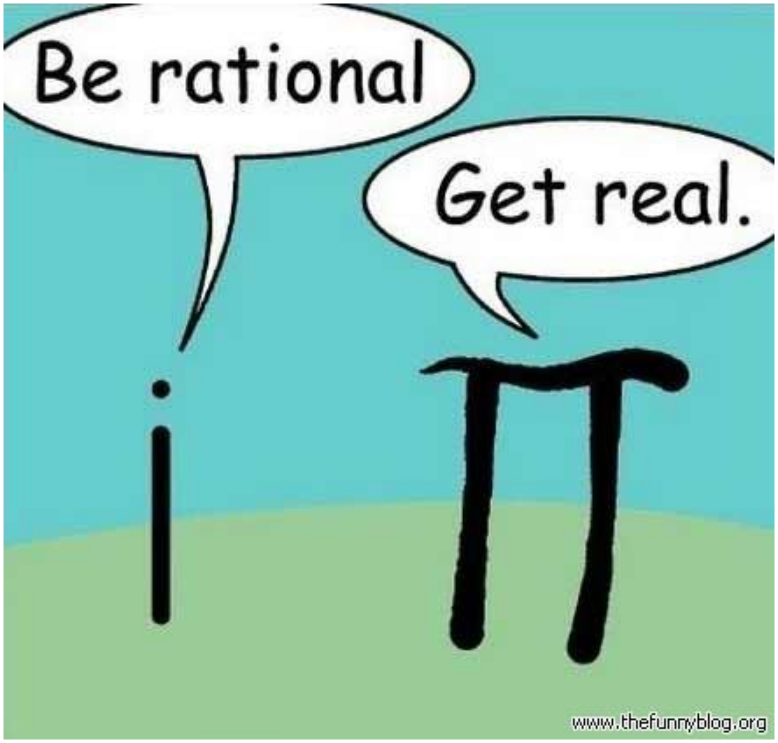
- Any number that can be written as an infinite decimal on the number line is real.

- In order to fully understand and invent real numbers, we would need a better understanding of infinity, limits, and calculus.
- Is $0.9999\cdots = 1$?

A: Yes
B: No
C: Depends on what infinity is
D: The question is ill-defined
E: None of the above

Imaginary numbers

- What about square roots of negative numbers?



- Any positive number squared is positive.
- Any negative number squared is positive.
- Thus, square roots of negative numbers can’t be either positive or negative, or on the real line at all.

New basic imaginary unit $i = \sqrt{-1}$

- Not on the real line, but we can imagine it as another imaginary line.

- And since it's a different direction, we often choose to represent it as perpendicular in the complex plane.

Complex numbers

- Complex numbers $a + bi$ are a combination of a real number and an imaginary number.

- Complex numbers are “complete” under taking square roots, addition, multiplication, etc.

Invention of complex numbers

A: Before 1000 BCE
B: 1000 BCE to 1000 CE
C: 1000 CE to 1500 CE
D: 1500 CE to 1800 CE
E: After 1800 CE