Clock arithmetic: Beyond counting… Lecture 6a: 2022-02-14

MAT A02 – Winter 2022 – UTSC

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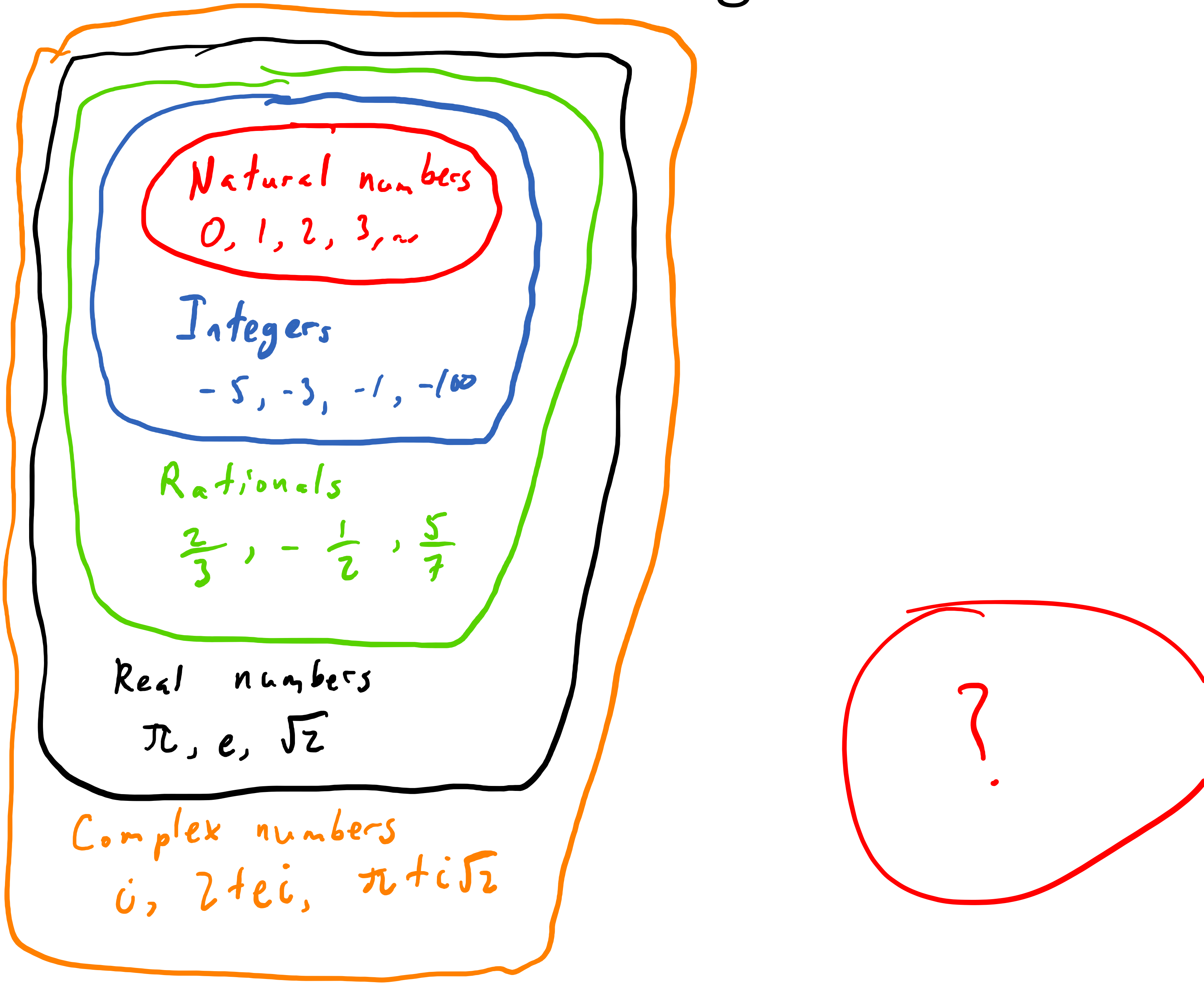
What do mathematicians do?

- Invent new ways of counting/measuring things
  - Addition
  - Multiplication
  - Exponentiation
- Figure out how to reverse operations
  - Subtraction
  - Division
  - Roots
- Invent new numbers to allow reversing stuff
  - Negative numbers
  - Fractions
  - Irrational numbers
  - Imaginary numbers

Extensions of counting numbers



Non-counting numbers?

- One way to think about numbers is that they are abstractions for counting objects.
- In this paradigm, operations like addition and multiplication are secondary consequences of better counting.

> What are other ways to think about math?
> Hint: think about reversing the above.

- Another way to think about math is as an abstract game where you have operations like “addition” and “multiplication”, and the numbers don’t mean anything, or mean something completely different.

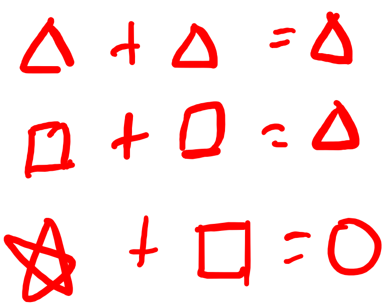
> What are some properties that you might want “addition” and “multiplication” to have?

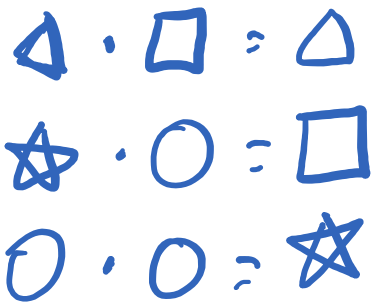
Completely arbitrary example

- Let's consider a number system with four numbers: {◁, □, ☆, ○)
- And the following addition/multiplication tables

+	◁	□	☆	○
◁	◁	□	☆	○
□	□	◁	○	☆
☆	☆	○	◁	□
○	○	☆	□	◁

×	◁	□	☆	○
◁	◁	◁	◁	◁
□	◁	□	☆	○
☆	◁	☆	○	□
○	◁	○	□	☆





Solve each of the following

+	◁	□	☆	○
◁	◁	□	☆	○
□	□	◁	○	☆
☆	☆	○	◁	□
○	○	☆	□	◁

×	◁	□	☆	○
◁	◁	◁	◁	◁
□	◁	□	☆	○
☆	◁	☆	○	□
○	◁	○	□	☆

- ☆+□ = ○
- (◁+ □)+ ○ = □ + ○ = ☆
- ☆×(☆× ○) = ☆ · □ = ☆
- ○ ×(☆+ ○) = ○ · □ = ○

A: ◁
B: □
C: ☆
D: ○
E: None of the above

Properties

+	◁	□	☆	○
◁	◁	□	☆	○
□	□	◁	○	☆
☆	☆	○	◁	□
○	○	☆	□	◁

×	◁	□	☆	○
◁	◁	◁	◁	◁
□	◁	□	☆	○
☆	◁	☆	○	□
○	◁	○	□	☆

- Commutative property:

$$x + y = y + x$$
$$x \times y = y \times x$$

Symmetry of tables

△ + □ = □ = □ + △

○ · ☆ = □ = ☆ · ○

- Associative property:

$$(x + y) + z = x + (y + z)$$
$$(x \times y) \times z = x \times (y \times z)$$

(△ + □) + ☆ = □ + ☆ = ○

△ + (□ + ☆) = △ + ○ = ○

- Identity properties:

$$0 + x = x$$
$$1 \times x = x$$

Notice △ + ? = ?

Notice □ · ? = ?

- Distributive property:

$$x \times (y + z) = xy + xz$$

Check by hand

Live demonstration

- Walking and counting steps.

- What about a Lego man on a globe?

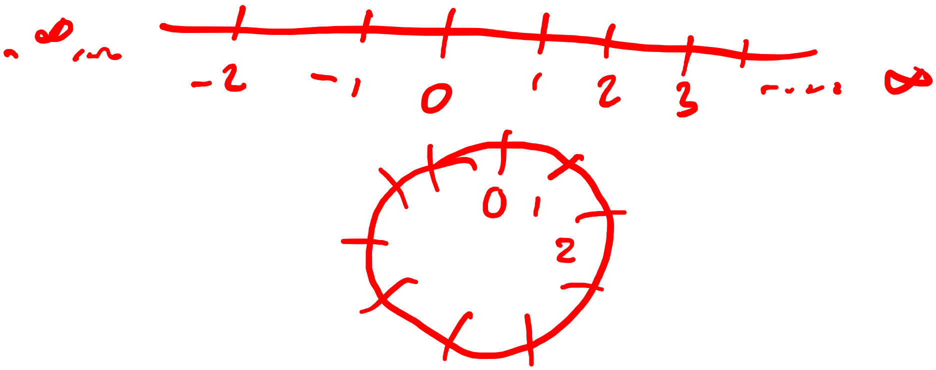
- What about on a donut shaped torus?

Blank slide

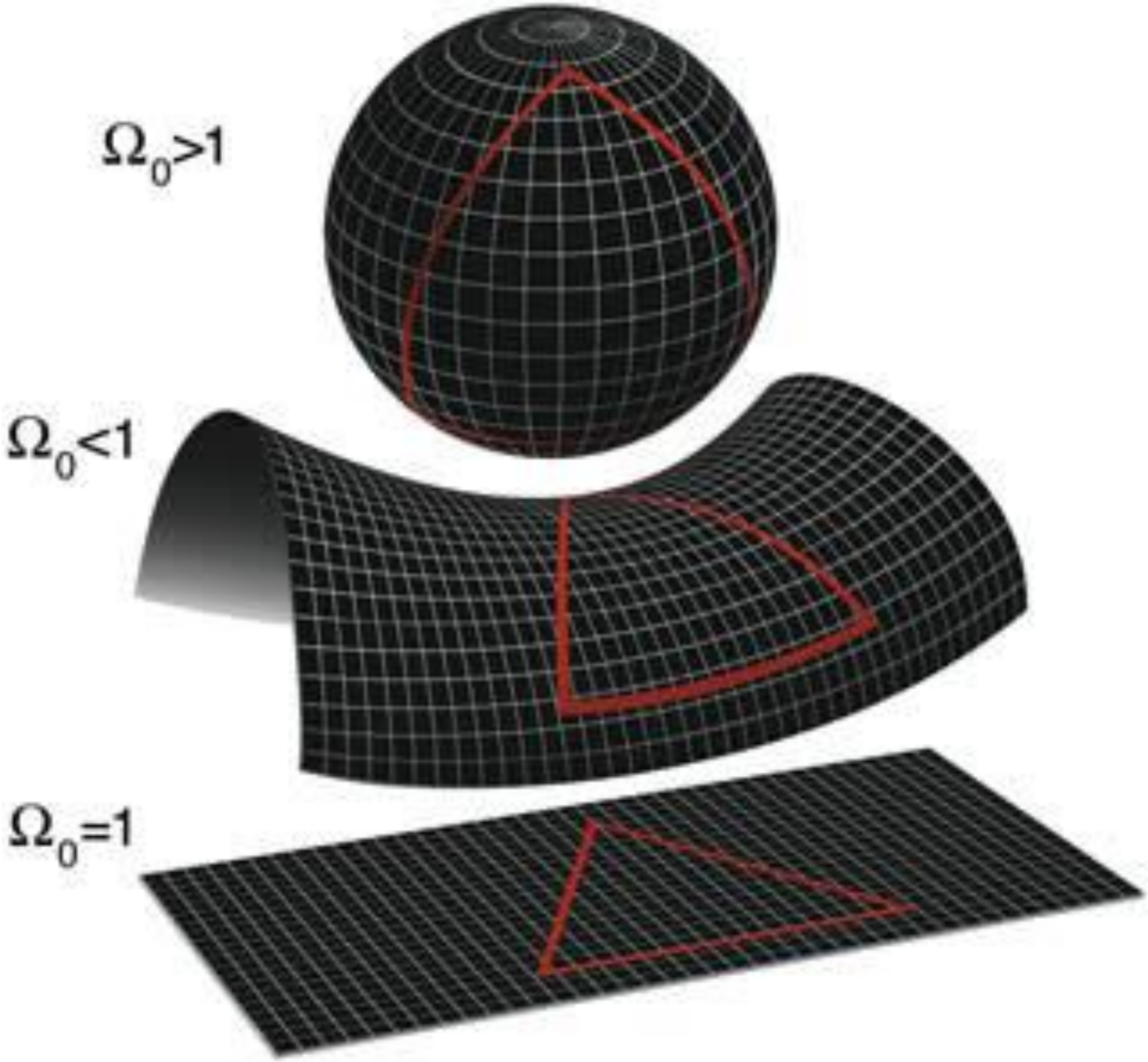
Clock arithmetic

- Sometimes, when you take enough steps forward, you end up back where you started.
- Notice, sometimes in real-world examples, you don't quite end up exactly where you started, like with time, but somewhere very similar.

Can you think of other examples where this happens?



Is the universe flat?



https://astronomy.com/news/2021/02/what-shape-is-the-universe