Roots in prime modulus arithmetic Lecture 9d: 2022-03-16

MAT A02 – Winter 2022 – UTSC

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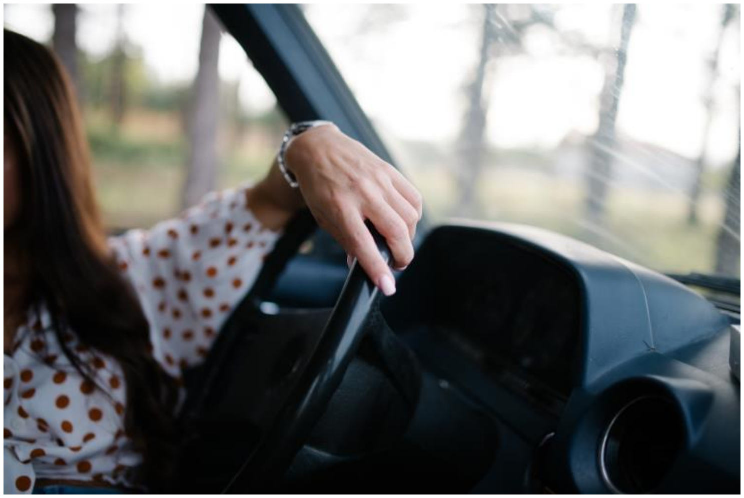
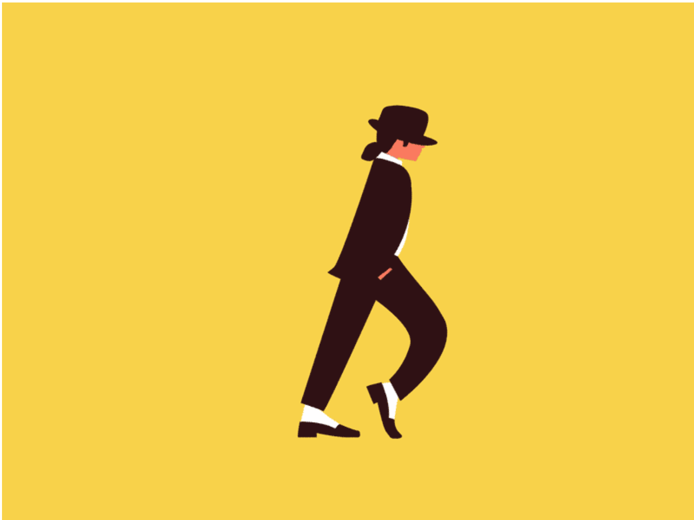
Reversing is hard

- We define addition, multiplication, exponentiation, etc.



https://www.flickr.com/photos/nenadstojkovic/50446472706/in/photostream/

- Subtraction, division, and roots, are reversing those operations and sometimes much harder.



Floris de Wit; https://dribbble.com/shots/5039546-Moonwalk

Division using multiplication table

mod 7

X	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

- Multiplication table encodes all pairs of products, so you can just look for the reverse.
- Example: $\frac{2}{5} \pmod 7$

Roots using powers table

mod 7

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	4	1	2	4	1	2	4	1	2	4	1	2
3	1	3	2	6	4	5	1	3	2	6	4	5	1	3
4	1	4	2	1	4	2	1	4	2	1	4	2	1	4
5	1	5	4	6	2	3	1	5	4	6	2	3	1	5
6	1	6	1	6	1	6	1	6	1	6	1	6	1	6

- A square root of a is a number b such that $b^2 \equiv a$.

- An kth root of a is a number b such that $b^k \equiv a$.

Roots using powers table

mod 7

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	4	1	2	4	1	2	4	1	2	4	1	2
3	1	3	2	6	4	5	1	3	2	6	4	5	1	3
4	1	4	2	1	4	2	1	4	2	1	4	2	1	4
5	1	5	4	6	2	3	1	5	4	6	2	3	1	5
6	1	6	1	6	1	6	1	6	1	6	1	6	1	6

- How many answers for each of the following?
- $\sqrt[3]{5}$
- $\sqrt[3]{6}$
- $\sqrt[5]{2}$
- $\sqrt[5]{3}$
- $\sqrt[13]{2}$

A: 0
B: 1
C: 2
D: 3
E: None of the above

Think like a mathematician

- When do kth roots exist in mod p arithmetic?
- When are kth roots unique? (only one root)

mod 7

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	4	1	2	4	1	2	4	1	2	4	1	2
3	1	3	2	6	4	5	1	3	2	6	4	5	1	3
4	1	4	2	1	4	2	1	4	2	1	4	2	1	4
5	1	5	4	6	2	3	1	5	4	6	2	3	1	5
6	1	6	1	6	1	6	1	6	1	6	1	6	1	6

Pattern recognition

- We can write out tables for small primes, look at all columns with all numbers, and try to find a pattern.
- Numbers k such that we can always find kth roots mod p:
  - Mod 5: 1, 3, 5, 7, 9, 11, 13, 15, …
  - Mod 7: 1, 5, 11, 13, 17, 19, 23, …
  - Mod 11: 1, 3, 7, 9, 13, 17, 19, 21, …
  - Mod 13: 1, 5, 7, 11, 13, 17, 19, 23, 25, …
- Can you spot the pattern?

A: Numbers are all odd numbers
B: Numbers are all prime numbers
C: Numbers are relatively prime to p
D: Numbers are relatively prime to $p - 1$
E: None of the above

Prime modulus facts (mod p)

- You can uniquely divide by any number except 0.

- Fermat's little theorem: $a^{p-1} \equiv 1 \pmod{p}$ if $a \not\equiv 0$.

Square roots

- In ordinary arithmetic, which of the following numbers is a square root of 1024? (without using a calculator?)

A: 25
B: 30
C: 32
D: 40
E: None of the above

- What if I told you $1024 = 2^{10}$? Then which of the following is a square root of 1024?

A: 5^2
B: $2 \cdot 3 \cdot 5$
C: 2^5
D: $2^3 \cdot 5$
E: None of the above

Square roots in mod 7

- In mod 7 arithmetic, what is the square root of 2?
- What if I told you $2 \equiv 1024 \equiv 2^{10}$? Then which of the following is a square root of 2?

A: 1
B: 2
C: 3
D: 4
E: None of the above

- What if I told you $2 \equiv 9 \equiv 3^2$? Then which of the following is a square root of 2?

A: 1
B: 2
C: 3
D: 4
E: None of the above

Higher roots

- In mod 7 arithmetic, what is the fifth root of 2?
- Strategy: use Fermat’s little theorem to find an equivalent of 2 as a power whose exponent is a multiple of 5.

Try it out

- In mod 7 arithmetic, what is a 5^{th} root of 3?

A: 2
B: 3
C: 4
D: 5
E: None of the above

Backwards reasoning for finding roots

- To solve $\sqrt[k]{a} \pmod p$, we need to find a number b such that $b^k \equiv a \pmod p$.

- One way to attempt this is to see if there exists a power m such that $b \equiv a^m$.

- That works precisely when $a^{mk} \equiv a \pmod p$

When does that strategy work?

- We need $a^{km} \equiv a \pmod{p}$.
- Or in other words, we need an exponent that is a multiple of k such that the two are equivalent.
- Fermat's Little Theorem says that
$$1 \equiv a^{(p-1)l}$$
$$a \equiv a^{(p-1)l+1}$$
- Equivalently, need to find integers m and l such that
$$mk = l(p-1) + 1$$
- We can rewrite this as:
$$1 = mk - l(p-1)$$
- Or, in other words, the strategy works if 1 is a combination of k and $p - 1$, which is true precisely when $\gcd(k, p-1) = 1$ (relatively prime)

One algorithm for $b \equiv \sqrt[k]{a} \bmod p$

- This algorithm works *if*
  - p is prime
  - $a \not\equiv 0 \bmod p$
  - k is relatively prime to $p - 1$
- Find $1 = mk - l(p - 1)$ using reverse Euclidean alg
- Then $\sqrt[k]{a} \equiv a^m$ mod p. Solve for $b \equiv a^m$ mod p.
- Check that $b^k \equiv a \pmod{p}$

Worked example

- $\sqrt[5]{10} \bmod 13$

Try it out

- Let p be prime, and $\gcd(k, p-1) = 1$.
- Given $b = \sqrt[k]{a} \pmod{p}$, find $1 = mk - l(p-1)$.
- Solution $b = a^m$
- Solve: $\sqrt[3]{6}$ mod 17

A: 2
B: 3
C: 4
D: 5
E: None of the above

Try it out

- Let p be prime, and $\gcd(k, p-1) = 1$.
- Given $b = \sqrt[k]{a} \pmod{p}$, find $1 = mk - l(p-1)$.
- Solution $b = a^m$
- Solve: $\sqrt[4]{6}$ mod 17

A: 2
B: 3
C: 4
D: 5
E: None of the above