Week 1: Integration

MAT A35 – Summer 2021 – UTSC

Prof. Yun William Yu

Survey of backgrounds

- Type in chat “a”, “b”, “c”, “d”, or “e” to state your field of interest:
  - A: Biology
  - B: Chemistry
  - C: Mathematics
  - D: Environmental Science
  - E: Other
- You may also add “?” to pump up the confusion meter to test out that functionality (or if you’re confused).

Re-invention of the trapezoid rule

A Mathematical Model for the Determination of Total Area Under Glucose Tolerance and Other Metabolic Curves

Mary M. Tai, MS, EDD

OBJECTIV areas under

RESEARC curve is co values on th areas can b total sum o Validity of model to th formulas wi metabolic cu

RESULTS — with precisio one or both intervals (abs 4) compare

CONCLUSIONS — The Tai model allows flexibility in experimental conditions, which means, in the case of the glucose-response curve, samples can be taken with differing time intervals and total area under the curve can still be determined with precision.

How long ago was the trapezoid rule invented?

A: Before 1 AD

B: 1-1000 AD

C: 1000-1500 AD

D: 1500-1900 AD

E: 1900-2000 AD

Estimation of total areas under curves of metabolic studies has become an increasingly popular tool for evaluating results from clinical trials as well as research investigations, such as total area under a glucose-tolerance or an energy-expenditure curve (1,2). Three formulas have been developed by Alder (3), Vecchio et al. (4), and Wolever et al. (5) to calculate the total area under a curve.

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However, except for Wolever et al.'s formula, other formulas tend to under- or overestimate the total area under a metabolic curve by a large margin.

RESEARCH DESIGN AND METHODS

Tai's mathematical model

Tai's model was developed to correct the

or overestimation of a metabolic curve. lows calculating the ith unequal units on tegy of this mathedivide the total area idividual small seg-, rectangles, and trin be precisely deterexisting geometric the individual seg- to obtain the total As shown in Fig. 1, xpressed as:

+ rectangle b +

+ triangle e +

+ rectangle h + . . .

th

^{2} $(x = y)$;

Let $X_1 = x_2 - x_1$; $X_2 = x_3 - x_2$

$X_3 = x_4 - x_3$; $X_4 = x_5 - x_4$;

$X_{n-1} = x_n - x_{n-1}$

$$\begin{aligned}\text{Total Area} &= \tfrac{1}{2}X_1(y_2 - y_1) + X_1y_1 + \tfrac{1}{2}X_2(y_3 - y_2) + X_2y_2 + \tfrac{1}{2}X_3(y_4 - y_3) + X_3y_3 \\ &+ \tfrac{1}{2}X_4(y_5 - y_4) + X_4y_4 + \ldots + \tfrac{1}{2}X_{n-1}(y_n - y_{n-1}) + X_{n-1}y_{n-1} \\ &= \tfrac{1}{2}(X_1y_1 + X_1y_2 + X_2y_2 + X_2y_3 + X_3y_3 + X_3y_4 + X_4y_4 + X_4y_5 + \ldots + X_{n-1}y_{n-1} + X_{n-1}y_n) \\ &= \tfrac{1}{2}[X_1(y_1 + y_2) + X_2(y_2 + y_3) + X_3(y_3 + y_4) + X_4(y_4 + y_5) + \ldots X_{n-1}(Y_{n-1} + Y_n)]\end{aligned}$$

If the curve passes the origin, $1/2[X_0y_1]$ should be added to above formula. If the curve intercepts at y_0 at the Y-axis, let $X_0 = x_1 - x_0$, $1/2[X_0(y_0 + y_1)]$ should be added to the above formula; Tai's formula applied to different conditions.

Ancient Babylonian astronomers calculated Jupiter's position from the area under a time-velocity graph

Mathieu Ossendrijver*

The idea of computing a body's displacement as an area in time-velocity space is usually traced back to 14th-century Europe. I show that in four ancient Babylonian cuneiform tablets, Jupiter's displacement along the ecliptic is computed as the area of a trapezoidal figure obtained by drawing its daily displacement against time. This interpretation is prompted by a newly discovered tablet on which the same computation is presented in an equivalent arithmetical formulation. The tablets date from 350 to 50 BCE. The trapezoid procedures offer the first evidence for the use of geometrical methods in Babylonian mathematical astronomy, which was thus far viewed as operating exclusively with arithmetical concepts.

The so-called trapezoid procedures examined in this paper have long puzzled historians of Babylonian astronomy. They belong to the corpus of Babylonian mathematical astronomy, which comprises about 450 tablets from Babylon and Uruk dating between 400 and 50 BCE. Approximately 340 of these tablets are tables with computed planetary or lunar data arranged in rows and columns (*1*). The remaining 110 tablets are procedure texts with computational instructions (*2*), mostly aimed at computing or verifying the tables. In all of these texts the zodiac, invented in Babylonia near the end of the fifth century BCE (*3*), is used as a coordinate system for computing celestial positions. The underlying algorithms are structured as branching chains of arithmetical operations (additions, subtractions, and multiplications) that can be represented as flow charts (*2*). Geometrical concepts are conspicuously absent from these texts, whereas they are very common in the Babylonian mathematical corpus (*4*–*7*). Currently four tablets, most likely written in Babylon between 350 and 50 BCE, are known to preserve portions of a trapezoid procedure (*8*). Of the four procedures, here labeled B to E (figs. S1 to S4), one (B) preserves a mention of Jupiter and three (B, C, E) are embedded in compendia of procedures dealing exclusively with Jupiter. The previously unpublished text D probably belongs to a similar compendium for Jupiter. In spite of these indications of a connection with Jupiter, their astronomical significance was previously not acknowledged or understood (*1*, *2*, *6*).

A recently discovered tablet containing an unpublished procedure text, here labeled text A (Fig. 1), sheds new light on the trapezoid procedures. Text A most likely originates from the same period and location (Babylon) as texts B to E (*8*). It contains a nearly complete set of instructions for Jupiter's motion along the ecliptic in accordance with the so-called scheme X.S$_1$ (*2*). Before the discovery of text A, this scheme was too fragmentarily known for identifying its connection with the trapezoid procedures. Covering one complete synodic cycle, scheme X.S$_1$ begins with Jupiter's heliacal rising (first visible rising at dawn), continuing with its first station (beginning of apparent retrograde motion), acronychal rising (last visible rising at dusk), second station (end of retrograde motion), and heliacal setting (last visible setting at dusk) (*2*). Scheme X.S$_1$ and the four trapezoid procedures are here shown to contain or imply mathematically equivalent descriptions of Jupiter's motion during the first 60 days after its first appearance. Whereas scheme X.S$_1$ employs a purely arithmetical terminology, the trapezoid procedures operate with geometrical entities.

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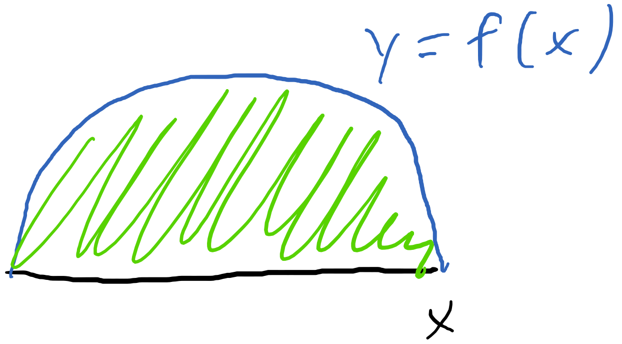
Differentiation and dimensional analysis

- Gets instantaneous rate of change of one variable vs another.
  - Let the variable x represent distance along the x-axis (in meters) and $y = f(x)$ be the height of the function on the y-axis (in meters).
    - Then $\frac{df}{dx} = f'(x)$ is the instantaneous slope (meters / meters = unitless).
  - Let the variable t represent time (in sec.) and the variable $x = x(t)$ be distanced traveled along the x-axis (in meters).
    - Then $\frac{dx}{dt} = \dot{x}(t)$ is the velocity (in meters/second)
  - Let the $v(t) = \dot{x}(t)$ be the velocity (meters/second, m/s)
    - Then $\frac{dv}{dt} = \ddot{x}(t)$ is the acceleration (in m/s^2)
  - Let $P(t)$ be the population (unitless count) at time t.
    - Then $\frac{dP}{dt}$ is the population growth rate (in number / second)

Integration

- “Opposite” of differentiation
  - “Summing up value as another variable changes”
  - Dimensional analysis: multiply together units along axes

Area:

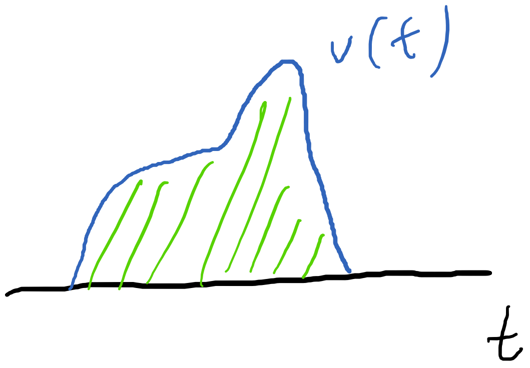


y-axis (meters)

x-axis (meters)

Area: m^2

Distance traveled:



velocity (m/s)

time (s)

Distance: (m)

Antiderivatives

$f(x) = 35x + 2021 \xrightarrow{\text{differentiate}} f'(x) = 35$

$f'(x) = 35 \xrightarrow[?]{\text{antidifferentiate}} \int f'(x)\,dx = \int 35\,dx$

$= 35x + 2021$?

But:

$$\left.\begin{aligned} f(x) &= 35x + \pi \\ f(x) &= 35x - \sqrt{2} \\ f(x) &= 35x + 5 \end{aligned}\right\} f'(x) = 35 \longrightarrow \int f'(x)\,dx$$

$= 35x + C$

Theorem: If two functions F and G have the same derivative $F'(x) = G'(x)$, then $F(x) = G(x) + C$, where C is a constant.

Can reverse many differentiation rules

Derivative rule	Integration rule
$\frac{d}{dx}[kx] = k$	$\int k\, dx = kx + C$
$\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r, \qquad r \neq -1$	$\int x^r\, dx = \frac{x^{r+1}}{r+1} + C, \qquad r \neq -1$
$\frac{d}{dx}[\ln\lvert x\rvert] = \frac{1}{\mathrm{x}} = \mathrm{x}^{-1}$	$\int x^{-1}\, dx = \ln\lvert x\rvert + C$
$\frac{d}{dx}\left[\frac{1}{a}e^{ax}\right] = e^{ax}$	$\int e^{ax}\, dx = \frac{1}{a}e^{ax} + C$
$\frac{d}{dx}\left[-\frac{1}{a}\cos ax\right] = \sin ax$	$\int \sin ax\ \, dx = -\frac{1}{a}\cos ax + C$
$\frac{d}{dx}\left[\frac{1}{\mathrm{a}}\sin ax\right] = \cos ax$	$\int \cos ax\ \, dx = \frac{1}{\mathrm{a}}\sin ax + C$
$\frac{d}{dx}\left[\frac{1}{\mathrm{a}}\tan ax\right] = \sec^2 ax$	$\int \sec^2 ax\ \, dx = \frac{1}{\mathrm{a}}\tan ax + C$
$\frac{d}{dx}\left[-\frac{1}{\mathrm{a}}\cot ax\right] = \csc^2 ax$	$\int \csc^2 ax\ \, dx = -\frac{1}{\mathrm{a}}\cot ax + C$
$\frac{d}{dx}\left[\frac{1}{a}\sec ax\right] = \sec ax \tan ax$	$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec x + C$
$\frac{d}{dx}\left[-\frac{1}{a}\csc ax\right] = \csc ax \cot ax$	$\int \csc ax \cot ax\, dx = -\frac{1}{\mathrm{a}}\csc ax + C$

Advanced rules

- Constant multiplication rule

$$\int kf(x)dx = k\int f(x)dx$$

- Addition/subtraction rule

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$
$$\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$$

- No product or quotient rule. See: Integration by Parts.

Examples

$$\int x^2 dx = \frac{1}{3}x^3 + C \qquad \frac{d}{dx}\left[\frac{1}{3}x^3 + C\right] = x^2$$

$$\int 4\sin x\, dx = 4\int \sin x\, dx = -4\cos x + C$$

$$\int \frac{2}{a}\, da = 2\ln|a| + C \qquad \int 5\,dy = 5y + C$$

$\int(ma + a^{35})da$, where m is a constant.

A: $m + 35a^{34}$

B: $m + 35a^{34} + C$

C: $\frac{m}{2}a^2 + \frac{1}{36}a^{36}$

D: $\frac{m}{2}a^2 + \frac{1}{36}a^{36} + C$

E: None of the above

$$\frac{d}{da}\left[\frac{m}{2}a^2 + \frac{1}{36}a^{36} + C\right]$$

$$= ma + a^{35} \quad \checkmark$$

Examples

$$\int \left[\cos 2x + 5e^{\pi x}\right] dx = \int \cos 2x \, dx + \int 5e^{\pi x} dx \quad \left(\begin{matrix}\text{addition}\\ \text{rule}\end{matrix}\right)$$

$$= \int \cos 2x \, dx + 5\int e^{\pi x} dx \quad \left(\begin{matrix}\text{multiply}\\ \text{by constant}\end{matrix}\right)$$

$$= \left[\frac{1}{2}\sin 2x + C_1\right] + 5\left[\frac{1}{\pi}e^{\pi x} + C_2\right]$$

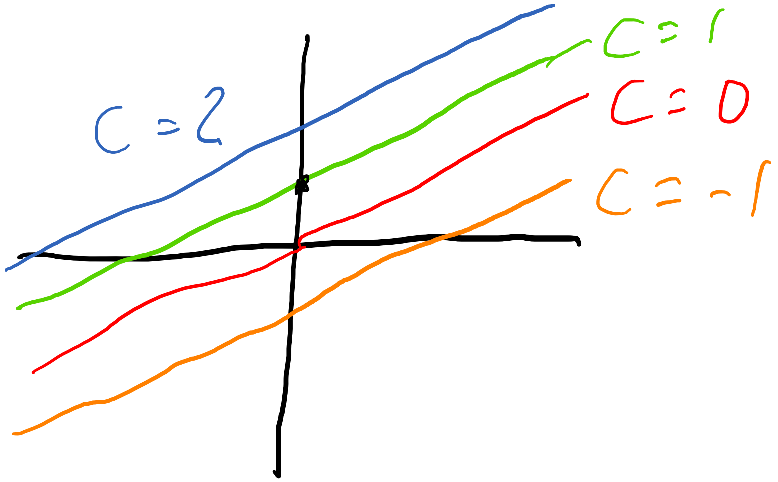
$$= \frac{1}{2}\sin 2x + \frac{5}{\pi}e^{\pi x} + C$$

Initial value problems and antiderivatives

- Recall that when there are infinitely many antiderivatives for a function. We can choose a specific function by giving an initial/boundary value.

Find the function f s.t. $f'(x) = \frac{1}{2}$ and $f(0) = 1$

$$\int f'(x)\,dx = \int \frac{1}{2}\,dx = \frac{x}{2} + C$$



$$f(x) = \frac{x}{2} + C$$

$$f(0) = \frac{0}{2} + C = 1$$

$$\Rightarrow C = 1$$

$$\boxed{f(x) = \frac{x}{2} + 1}$$

Example - IVP

- Consider the function f such that $f'(x) = x^2 + 2x$ where $f(0) = 0$.

What is $f(1)$?

A: $\frac{4}{3}$
B: $\frac{2}{3}$
C: 1
D: $x^2 + C$
E: None of the above

integration rules

$$f(x) = \int (x^2 + 2x)\,dx = \frac{1}{3}x^3 + x^2 + C$$

$$f(0) = 0 = \frac{1}{3}\cdot 0^3 + 0^2 + C$$

$$\Rightarrow C = 0$$

$$f(x) = \frac{1}{3}x^3 + x^2 \cancel{+C}$$

$$f(1) = \frac{1}{3}\cdot 1^3 + 1^2 = \frac{4}{3}$$

$x=1$