Week 1: Integration
Lecture 3: 2021-05-12

MAT A35 – Summer 2021 – UTSC

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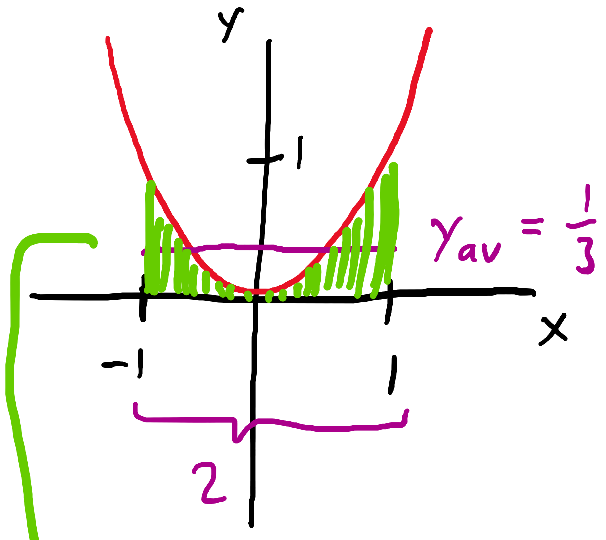
Average of a function

- Let $f : [\mathrm{a}, \mathrm{b}] \to \mathbb{R}$ be a continuous function. Then its average value $y_{av} = \frac{1}{b-a}\int_a^b f(x)dx$.

Ex. $f(x) = x^2$. Find the average value between -1 and 1.

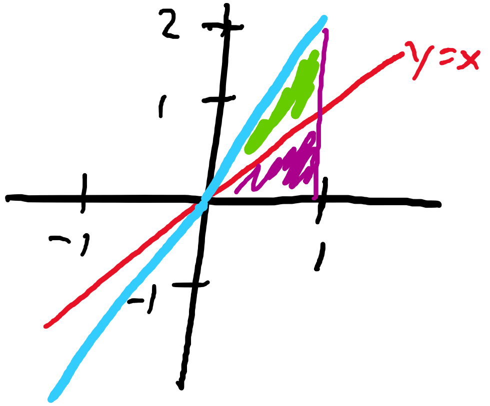


Total area $= \frac{2}{3}$

$$y_{av} = \frac{1}{\underbrace{1-(-1)}_{2}} \int_{-1}^{1} x^2 \, dx = \frac{1}{2}\left[\frac{1}{3}x^3\right]_{-1}^{1}$$

$$= \frac{1}{6}\left[1^3 - (-1)^3\right] = \frac{1}{3}$$

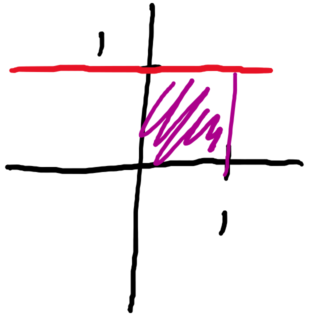
Properties of definite integrals

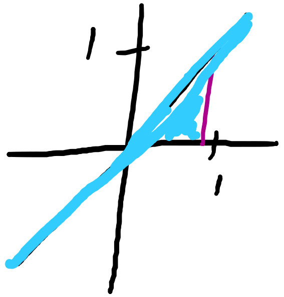


- Constant multiplication: $\int_a^b k \cdot f(x)dx = k \cdot \int_a^b f(x)dx$

$$\int_0^1 2 \cdot x \, dx = 2 \int_0^1 x \, dx$$

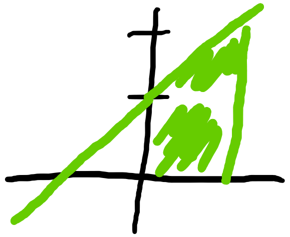
- Sum of different integrands with same bounds
  - $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

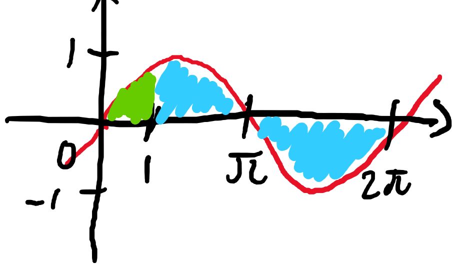




$$\int_0^1 (1+x)dx = \int_0^1 1\,dx + \int_0^1 x\,dx$$

- Sum of same integrand with touching bounds
  - $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$ where $a < b < c$





$$\int_0^1 \sin x\,dx + \int_1^{2\pi} \sin x\,dx = \int_0^{2\pi} \sin x\,dx$$

Try it out

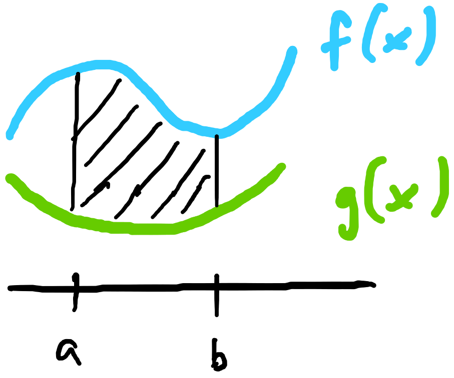
$$\underbrace{\int_0^2 x^2 dx + \int_0^2 5dx}_{\int_0^2 (x^2+5)dx} + \underbrace{\int_1^3 (x^2+5)dx - \int_1^2 (x^2+5)dx}_{\int_2^3 (x^2+5)dx}$$

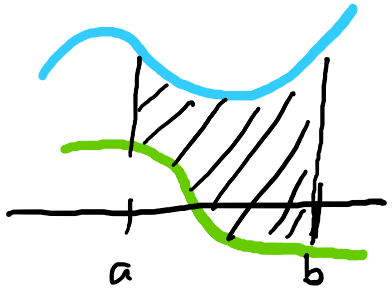
$$= \int_0^3 (x^2+5)dx = \left[\frac{1}{3}x^3+5x\right]_0^3 = 9+15=24$$

A: 24
B: 27
C: 30
D: 33
E: None of the above

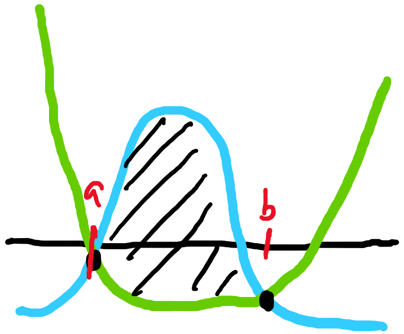
Area between curves

Let f and g be continuous functions, and suppose that $f(x) \geq g(x)$ over the interval $[a, b]$. Then the area of the region between the two curves on that interval is $\int_a^b [f(x) - g(x)]dx$.





doesn't matter that g(x) goes negative



When $[a, b]$ are unknown, can compute the intersection points to figure out the area bounded by curves.

Example

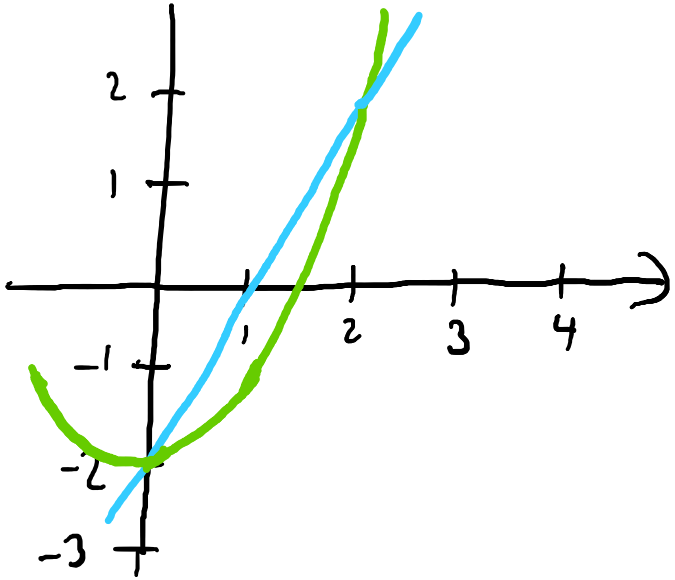
- Find the area bounded by the graphs of $f(x) = 2x - 2$ and $g(x) = x^2 - 2$.

Intersections when $f(x) = g(x)$ $\Rightarrow$ $x^2 - 2 = 2x - 2$

$\Rightarrow$ $x^2 - 2x = 0$

$\Rightarrow$ $x(x-2) = 0$

$x = 0, 2$



$$\text{Area} = \int_0^2 [f(x) - g(x)]\,dx = \int_0^2 (2x - x^2)\,dx$$

$$= \left[x^2 - \frac{1}{3}x^3\right]_0^2 = 4 - \frac{8}{3} = \underline{\underline{\frac{4}{3}}}.$$

Try it out

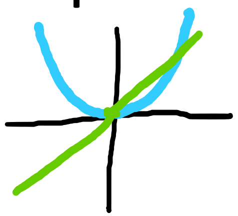
- Find the area bounded by graphs of $f(x) = x^2$ and $g(x) = x$.
- Step 1: find the intersection points.

A: -1, 1
B: 0, 2
C: -1, 0
D: 0, 1
E: None of the above

$x^2 = x$
$\Rightarrow x^2 - x = 0$
$x(x-1) = 0$
$\Rightarrow x = 0, 1$

- Step 2: Decide which graph is on top.

A: $f(x)$
B: $g(x)$
C: neither
D: both
E: ?????



$x > x^2$ for $x \in [0,1]$.
Also can check $f(0.5) = 0.25 < g(0.5) = 0.5$

- Step 3: Compute the integral.

A: 1/3
B: 1/4
C: 1/5
D: 1/6
E: None of the above

$$\int_0^1 [x - x^2]\,dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Chain rule → Substitution rule

- Chain rule: Let $f = f(u)$ be a function of u and $u = u(x)$ be a function of x. Then $\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$.

Ex. $\frac{d}{dx}\left[(2x+1)^2\right]$ Let $f(u) = u^2$, $u(x) = 2x+1$

$\frac{df}{du} = 2u$ $\frac{du}{dx} = 2$

$= 2u \cdot 2$

$= 2(2x+1) \cdot 2 = 4(2x+1).$

- "u-substitution" is the opposite of the chain rule.

Ex. $\int 4(2x+1)\,dx$ Let $u = 2x+1$

$du = 2dx \Rightarrow dx = \frac{1}{2}du$

$= \int 4u \cdot \frac{1}{2}du = \int 2u\,du$

$= u^2 + C = (2x+1)^2 + C = 4x^2 + 4x + 1 + C = 4x^2 + 4x + C$

Substitution rule algorithm

- Step 1: Guess an appropriate u
- Step 2: Compute du, dx, and x
- Step 3: Substitute in to get rid of all the x's
- Step 4: Integrate as a function of u
- Step 5: Convert back to x's

$$\int 2xe^{x^2}\,dx$$

1. Let $u = x^2$
2. $du = 2x\,dx$

hyperreal infinitesimals

$$\left(\frac{d}{dx}u = 2x \qquad \frac{du}{dx} = 2x \qquad du = 2x\,dx \right)$$

3. $$\int 2xe^{x^2}\,dx = \int e^u\,du$$

4. $$= e^u + C$$

5. $$= e^{x^2} + C$$

Check

$$\frac{d}{dx}\left[e^{x^2}\right] = 2xe^{x^2}$$

Example

$$\int \frac{1}{1+x}\,dx \qquad \text{Let } u = 1+x \qquad \frac{du}{dx} = 1$$

$$du = dx \qquad du = dx$$

$$\int \frac{1}{u}\,du = \ln|u| + C = \ln|1+x| + C$$

$$\int 4x\sqrt{x^2+1}\,dx. \qquad \text{Let } u = x^2+1$$

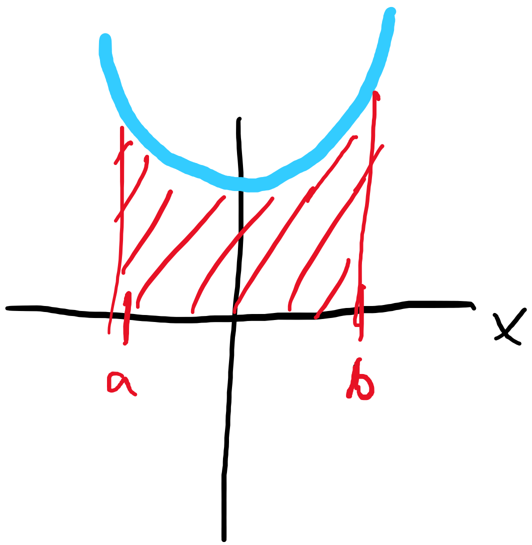
$$du = 2x\,dx$$

$$= \int 2\sqrt{u}\,du$$

$$= \int 2u^{\frac{1}{2}}\,du = \frac{2}{3}\cdot 2\,u^{\frac{3}{2}} + C = \frac{4}{3}u^{\frac{3}{2}} + C = \frac{4}{3}(x^2+1)^{\frac{3}{2}} + C$$

Substitution for definite integrals

$$\int_a^b (1+x^2)\,2x\,dx = \int_{x=a}^{x=b} (1+x^2)\,2x\,dx$$



Let $u = 1+x^2$

$du = 2x\,dx$

$$= \int_{u=1+a^2}^{u=1+b^2} u\,du = \left[\frac{1}{2}u^2\right]_{u=1+a^2}^{u=1+b^2}$$

$$= \left[\frac{1}{2}(1+b^2)^2\right] - \left[\frac{1}{2}(1+a^2)^2\right]$$

Need to change limits of integration, but don't convert back to x's.

Try it out

- $\int_0^2 \frac{x}{(1+x^2)^2}\,dx$ Let $u = 1+x^2$, $du = 2x\,dx$

$$= \int_{u=1}^{u=5} \frac{1}{2}\cdot\frac{1}{u^2}\,du = \frac{1}{2}\left[-\frac{1}{u}\right]\Bigg|_{u=1}^{u=5}$$

$$= \frac{1}{2}\left[-\frac{1}{5}+1\right] = \frac{1}{2}\cdot\frac{4}{5} = \frac{2}{5}$$

A: 0
B: 0.2
C: 0.4
D: 0.6
E: None of the above

- $\int \tan x\,dx$. Hint: $\tan x = \frac{\sin x}{\cos x}$. Let $u = \cos x$

$du = -\sin x\,dx$

$$\int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{u}\,du$$

$$= -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

A: $\ln|\sin x|^2 + C$
B: $-\ln|\sin x| + C$
C: $\ln|\cos x|^2 + C$
D: $-\ln|\cos x| + C$
E: None of the above

Integration techniques – partial fractions

- Sometimes, it is easier to integrate if you break up a complicated expression into several simpler ones. One way to do this is with a partial fractions decomposition:

$$\frac{h(x)}{f(x)g(x)} = \frac{A(x)}{f(x)} + \frac{B(x)}{g(x)}$$

Where $h(x), f(x), g(x), A(x), B(x)$ are all polynomials in x.

Ex. $\frac{1}{1-x^2} = \frac{1}{(1+x)(1-x)} = \frac{A}{1+x} + \frac{B}{1-x}$

Need: $A(1-x) + B(1+x) = 1$

$\Rightarrow A + B + x(-A+B) = 1$

$\Rightarrow A+B=1, \quad -A+B=0$

$\Rightarrow A = \frac{1}{2}, \quad B = \frac{1}{2}$

$\Rightarrow \frac{1}{1-x^2} = \frac{\frac{1}{2}}{1+x} + \frac{\frac{1}{2}}{1-x}$

Example

$$\int \frac{1}{1-x^2}\,dx = \int \left[\frac{1}{2}\cdot\frac{1}{1-x} + \frac{1}{2}\cdot\frac{1}{1+x}\right] dx$$

$$= \frac{1}{2}\int \frac{1}{1-x}\,dx + \frac{1}{2}\int \frac{1}{1+x}\,dx = \frac{1}{2}\Big[\ln|x+1| - \ln|x-1|\Big] + C$$

$$\int \frac{1}{1-x}\,dx = -\int \frac{1}{x-1}\,dx$$

$$\boxed{\begin{aligned}\text{Let } u &= x-1\\ du &= dx\end{aligned}}$$

$$= -\int \frac{1}{u}\,du$$

$$= -\ln|u| + C$$

$$= -\ln|x-1| + C$$

$$\int \frac{1}{1+x}\,dx = \int \frac{1}{u}\,du$$

$$\boxed{\begin{aligned}\text{Let } u &= x+1\\ du &= dx\end{aligned}}$$

$$= \ln|u| + C$$

$$= \ln|x+1| + C$$

Try it out: $\int \frac{5x+1}{2x^2-x-1} dx$

1: Factor: $2x^2 - x - 1$

FOIL $= (2x+1)(x-1)$

or

$x = \frac{1 \pm \sqrt{1+8}}{4} = 1, -\frac{1}{2}$

$\Rightarrow C(x-1)(x+\frac{1}{2})$

$\hookrightarrow = 2$

A: $(2x-1)(x-1)$
B: $(2x+1)(x-1)$
C: $(2x-1)(x+1)$
D: $(2x+1)(x+1)$
E: None of the above

2: Solve for $\frac{5x+1}{2x^2-x-1} = \frac{A}{2x+1} + \frac{B}{x-1}$

$A(x-1) + B(2x+1) = 5x+1$

$x(A+2B) + (-A+B) = 5x+1$

$\begin{cases} A+2B=5 \\ -A+B=1 \end{cases} \Rightarrow \begin{matrix} 3B=6 \\ B=2 \end{matrix} \Rightarrow A=1$

A: $A=1, B=1$
B: $A=1, B=2$
C: $A=2, B=2$
D: $A=2, B=1$
E: None of the above

3: Integrate $\int \frac{5x+1}{2x^2-x-1} dx = \int \frac{1}{2x+1} dx + \int \frac{2}{x-1} dx$

$= \frac{1}{2} \ln|2x+1| + 2 \ln|x-1| + C$

Product Rule → Integration by parts

- Recall $\frac{d}{dx}[u(x)v(x)] = u(x)v'(x) + u'(x)v(x)$

Ex. $\frac{d}{dx}\left[(x+1)e^x\right] = (x+1)e^x + e^x = xe^x + 2e^x$

- Integration by parts is the opposite of the product rule:
  - $\frac{d}{dx}[u(x)v(x)] = u(x)v'(x) + u'(x)v(x) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$
  - $d[u(x)v(x)] = u \cdot dv + v \cdot du$
  - $u \cdot dv = d[u(x)v(x)] - v \cdot du$
  - $\int u \cdot dv = \int d[u(x)v(x)] - \int v \cdot du$
  - $\int u\, dv = uv - \int v\, du$

Integration by parts algorithm

- $\int u\, dv = uv - \int v\, du$
- Step 1: Guess which part is u and which part is dv
- Step 2: Apply the formula above and hope you can solve $\int v\, du$
- Step 3: If it doesn't, try again with a different guess for u and dv.

- Step ?: Give up if no guess seems to work. The integral might not be amenable to integration by parts.

Example ($\int u\,dv = uv - \int v\,du$)

$$\int \underbrace{\ln x}_{u} \; \underbrace{dx}_{dv}$$

$u = \ln x \qquad v = x$

$du = \frac{1}{x}\,dx \qquad dv = dx$

$$= (\ln x)\cdot x - \underbrace{\int x \cdot \frac{1}{x}\,dx}_{\int dx \,= x + C}$$

$$= x \ln x - x + C$$

Example ($\int u\,dv = uv - \int v\,du$)

$\int x \ln x \, dx$

Guess 1: $u = 1$ $v = \int x \ln x \, dx$ ✗

$du = 0$ $dv = x \ln x \, dx$

Guess 2: $u = x \ln x$ $v = x$

$du = (1 + \ln x)dx$ $dv = dx$

$$\int x \ln x \, dx = x^2 \ln x - \underbrace{\int x(1+\ln x)\,dx}_{\int [x + x\ln x]\,dx}$$

✗

Guess 3: $u = \ln x$ $v = \frac{1}{2}x^2$

$du = \frac{1}{x}dx$ $dv = x\,dx$

$$= \frac{x^2}{2} \cdot \ln x - \underbrace{\int \frac{1}{2}x^2 \cdot \frac{1}{x}\,dx}$$

$$\int \frac{1}{2} x \, dx = \frac{1}{4}x^2 + C$$

$$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$$

$\int u\,dv = uv - \int v\,du$

Try it out: $\int x^2 e^x dx$

Let $u = x^2$ $\quad v = e^x$

$du = 2x\,dx$ $\quad dv = e^x dx$

$= x^2 e^x - 2\underbrace{\int x e^x dx}$

Let $u = x$ $\quad v = e^x$

$du = dx$ $\quad dv = e^x dx$

$= x^2 e^x - 2\left[x e^x - \int e^x dx\right]$

$= x^2 e^x - 2xe^x + 2e^x$

Write your answer in chat.