Systems of Linear ODEs Lecture 10a: 2021-07-22

MAT A35 – Summer 2021 – UTSC

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System of two first-order ODEs

- Let t be the independent time variable, and let x and y be two dependent variables.
- If a, b, c, d are constants, then $\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases}$ $\qquad \dot{x} = \dfrac{dx}{dt} \quad \dot{y} = \dfrac{dy}{dt}$

  is a homogeneous system of two first-order linear ODEs.
- Similarly, $\begin{cases} \dfrac{dx}{dt} = ax + by + f(t) \\ \dfrac{dy}{dt} = cx + dy + g(t) \end{cases}$

  is an inhomogeneous system of two first-order linear ODEs.

Reduction method

- We can solve a system of two first-order linear ODEs by converting it into a single second-order linear ODE.
  - First compute $\ddot{x}$ as a function of $x, y, \dot{y}$ from the first equation.
  - Then eliminate $\dot{y}$ by substituting in the second equation.
  - Then, eliminate y by substituting in the first equation.

Ex. $\begin{cases} \dot{x} = \dfrac{dx}{dt} = x + 2y \\ \dot{y} = \dfrac{dy}{dt} = 4x - y \end{cases}$ $\Rightarrow y = \dfrac{\dot{x} - x}{2}$

$\dfrac{d}{dt}$

$\dfrac{d}{dt}\dfrac{dx}{dt} = \dfrac{d^2x}{dt^2} = \ddot{x}$

$\ddot{x} = \dot{x} + 2\dot{y}$

$\ddot{x} = \dot{x} + 2(4x - y)$

$\ddot{x} = \dot{x} + 8x - 2y$

$\ddot{x} = \dot{x} + 8x - 2\left(\dfrac{\dot{x} - x}{2}\right)$

$\ddot{x} = \dot{x} + 8x - \dot{x} + x \Rightarrow \ddot{x} = 9x$

$\Rightarrow \quad \ddot{x} - 9x = 0$

Solve 2nd-order ODE

- Find eigenvalues/roots of the characteristic equation.
- Solve for x.
- Solve for y.

$$\begin{cases} \ddot{x} - 9x = 0 \\ \lambda^2 - 9 = 0 \\ \lambda = \pm 3 \end{cases}$$

$$x = c_1 e^{3t} + c_2 e^{-3t}$$

$$y = \frac{\dot{x} - x}{2} \qquad \dot{x} = 3c_1 e^{3t} - 3c_2 e^{-3t}$$

$$y = \frac{1}{2}\left[3c_1 e^{3t} - 3c_2 e^{-3t} - \left(c_1 e^{3t} + c_2 e^{-3t}\right)\right]$$

$$\boxed{x = c_1 e^{3t} + c_2 e^{-3t} \qquad y = c_1 e^{3t} - 2c_2 e^{-3t}}$$

general solution

Initial values

- If there are initial values, plug them in.

Ex. $x = c_1 e^{3t} + c_2 e^{-3t}$ $\quad y = c_1 e^{3t} - 2c_2 e^{-3t}$

$\boxed{\text{IVP:}}$ $x(0) = 2$ $\quad y(0) = -1$

$2 = c_1 e^{0} + c_2 e^{0}$ $\quad -1 = c_1 e^{0} - 2c_2 e^{0}$

$2 = c_1 + c_2$ $\quad -1 = c_1 - 2c_2$

$\Rightarrow 3 = 3c_2 \Rightarrow c_2 = 1 \Rightarrow c_1 = 1$

$\Rightarrow x = e^{3t} + e^{-3t}$, $y = -e^{3t} - 2e^{-3t}$

Try it out

$\frac{d}{dt}$ $\dot{x} = \frac{d}{dt} y$

- $\begin{cases} \boxed{\dot{x} = y} \\ \dot{y} = -x + 2y \end{cases}$ $x(0) = 1, y(0) = 3$
- Step 1: Find $\ddot{x}$.

$\ddot{x} = \dot{y}$

A: $\ddot{x} = 0$
B: $\ddot{x} = 1$
C: $\ddot{x} = y$
D: $\ddot{x} = \dot{y}$
E: None of the above

- Step 2: Get rid of $\dot{y}$.

$\ddot{x} = \dot{y} = -x + 2y$

A: $\ddot{x} = y$
B: $\ddot{x} = -x + 2y$
C: $\ddot{x} = -xy + 2y^2$
D: $\ddot{x} = \dot{y}$
E: None of the above

- Step 3: Get rid of y and rewrite.

$\ddot{x} = -x + 2\dot{x}$
$\ddot{x} - 2\dot{x} + x = 0$

A: $\ddot{x} + 2\dot{x} + x = 0$
B: $\ddot{x} - 2\dot{x} + x = 0$
C: $\ddot{x} + 2\dot{x} - x = 0$
D: $\ddot{x} - 2\dot{x} - x = 0$
E: None of the above

Try it out (continued)

$c_1 e^t + c_2 t e^t$

- Step 4: Solve the ODE

$\ddot{x} - 2\dot{x} + x = 0$

$\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$

$\lambda = 1$, mult 2

$\Rightarrow x = c_1 e^t + c_2 t e^t$

A: $x = c_1 e^t + c_2 e^{-t}$
B: $x = c_1 e^t + c_2 e^{2t}$
C: $x = c_1 e^t + c_2 x e^t$
D: $x = c_1 e^t + c_2 t e^t$
E: None of the above

- Step 5: Plug back in to solve for y.

$y = \dot{x}$

$y = c_1 e^t + \frac{d}{dt}\left[c_2 t e^t\right]$ product rule

$y = c_1 e^t + c_2 e^t + c_2 t e^t$

$y = (c_1 + c_2) e^t + c_2 t e^t$

A: $y = c_1 e^t + c_2 t e^t$
B: $y = (c_1 + c_2) e^t + c_2 t e^t$
C: $y = c_1 e^t + (c_1 + c_2) t e^t$
D: $y = (c_1 + c_2) e^t + (c_2 - c_2) t e^t$
E: None of the above

Try it out (continued)

- Plug in initial values $x(0) = 1, y(0) = 3$

$$x = c_1 e^t + c_2 t e^t$$

$$y = (c_1 + c_2) e^t + c_2 t e^t$$

$$1 = x(0) = c_1 e^0 + c_2 \cdot 0 e^0 = c_1$$

$$\Rightarrow c_1 = 1$$

$$3 = y(0) = c_1 + c_2$$

$$\Rightarrow c_2 = 2$$

A: $c_1 = 1, c_2 = 1$
B: $c_1 = 1, c_2 = 2$
C: $c_1 = 2, c_2 = 1$
D: $c_1 = 2, c_2 = 2$
E: None of the above

$$\boxed{\begin{aligned} x &= e^t + 2te^t \\ y &= 3e^t + 2te^t \end{aligned}}$$

Inhomogeneous example

- $\begin{cases} \dot{x} = x + y + 9t \\ \dot{y} = 4x + y + 3 \end{cases}$ $\quad y = \dot{x} - x - 9t$

1. $\ddot{x} = \dot{x} + \dot{y} + 9$ $\quad$ Take $\frac{d}{dt}\dot{x}$

2. $\ddot{x} = \dot{x} + 4x + y + 12$ $\quad$ Subst. in $\dot{y}$

3. $\ddot{x} = \dot{x} + 4x + (\dot{x} - x - 9t) + 12$ $\quad$ Subst in y

$$\ddot{x} = 2\dot{x} + 3x - 9t + 12$$

$$\ddot{x} - 2\dot{x} - 3x = -9t + 12$$

Rewrite

Example (continued)

- $\ddot{x} - 2\dot{x} - 3x = -9t + 12$

$\lambda^2 - 2\lambda - 3 = 0$

$(\lambda - 3)(\lambda + 1) = 0$

$\lambda = -1, 3$

$x_h = c_1 e^{-t} + c_2 e^{3t}$

Ansatz

$x_p = At + B$

$\dot{x}_p = A$

$\ddot{x}_p = 0$

$0 - 2A - 3(At + B) = -9t + 12$

$-3At + (-2A - 3B) = -9t + 12$

$-3A = -9 \Rightarrow A = 3$

$-2A - 3B = 12 \Rightarrow B = -6$

$\Rightarrow x_p = 3t - 6$

$x_{gen} = c_1 e^{-t} + c_2 e^{3t} + 3t - 6$

Solve for y

$y = \dot{x} - x - 9t$

- $\begin{cases} \dot{x} = x + y + 9t \\ \dot{y} = 4x + y + 3 \end{cases}$
- $x = c_1 e^{-t} + c_2 e^{3t} + 3t - 6$

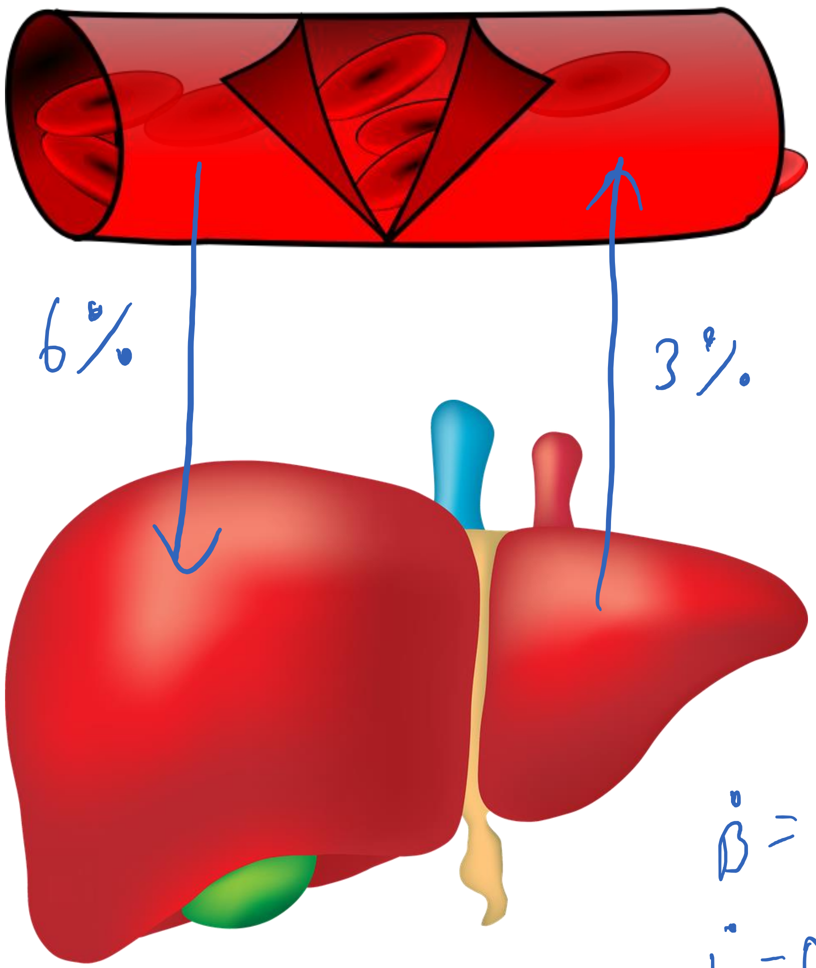
$\dot{x} = -c_1 e^{-t} + 3c_2 e^{3t} + 3$

$y = -c_1 e^{-t} + 3c_2 e^{3t} + 3 - \left(c_1 e^{-t} + c_2 e^{3t} + 3t - 6\right) - 9t$

$y = -2c_1 e^{-t} + 2c_2 e^{3t} - 12t + 9$

Application (Bittinger, 9.3, Ex. 5)

- 3mg of glactosyl human serum albumin (Tc-GSA) is injected into a patient's bloodstream to measure liver function. After injection, Tc-GSA is transferred from the blood to the liver at 6% per minute, and from the liver into blood at 3% per minute.
- How much Tc-GSA is in the blood or liver as a function of time?



$B(0) = 3\,mg$

$L(0) = 0\,mg$

$$\begin{array}{c} \boxed{B} \\ 0.06B \downarrow \quad \uparrow 0.03L \\ \boxed{L} \end{array}$$

$\dot{B} = -0.06B + 0.03L$

$\dot{L} = 0.06B - 0.03L$

Application (continued)

- $\begin{cases} \dot{B} = -0.06B + 0.03L \\ \dot{L} = \ \ 0.06B - 0.03L \end{cases}$

Application (continued)

- $\ddot{B} + 0.09\dot{B} = 0$

$$L = \frac{100}{3}\dot{B} + 2B$$

Initial Value Problem

- $\begin{cases} B(t) = c_1 + c_2 e^{-0.09t} \\ L(t) = 2c_1 - c_2 e^{-0.09t} \end{cases}$, with initial values $\begin{cases} B(0) = 3 \\ L(0) = 0 \end{cases}$