# Matrix ODE representations Lecture 10b: 2021-07-28

MAT A35 – Summer 2021 – UTSC

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# Homogeneous system as matrix equation

- We can rewrite a homogeneous system of 1st-order ODEs as a matrix-vector ODE.
- $\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases} \rightarrow \underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}}_{\dot{z}} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{z}$

$\dot{z} = A z$

Ex. $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} x \\ -4x + 2y \end{bmatrix}$

$\begin{cases} \dot{x} = x \\ \dot{y} = -4x + 2y \end{cases}$

$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$

- The matrix-form equation $\dot{z} = Az$ has solutions $z(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

Ex. $z = \begin{bmatrix} e^t \\ 4e^t - 2e^{2t} \end{bmatrix}$ $\quad \dot{z} = A z$

$z = \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad x = e^t$, $y = 4e^t - 2e^{2t}$

$\begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} e^t \\ 4e^t - 2e^{2t} \end{bmatrix}$

$= \begin{bmatrix} e^t \\ -4e^t + 8e^t - 4e^{2t} \end{bmatrix} = \begin{bmatrix} e^t \\ 4e^t - 4e^{2t} \end{bmatrix}$

# Solution to a matrix-form equation

$\lambda v = Av$

$0 = Av - \lambda v$

$\lambda v - Av = 0$

$(\lambda I - A) = 0$

- If $y' = ky$, then $y = c_0 e^{kx}$ for single-variable ODEs.
- If $\dot{z} = Az$, then perhaps $z = e^{\lambda t} v$, for some constant $\lambda$ and vector $v$.

$\dot{z} = \lambda e^{\lambda t} \cdot v$

$\lambda e^{\lambda t} \cdot v = A(e^{\lambda t} \cdot v)$

$e^{\lambda t}(\lambda v) = e^{\lambda t}(Av)$

$\lambda v = Av$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{z} = \underbrace{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}_{2z}$$

$\lambda = 2,\ v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda - 1 \end{vmatrix} = 0$$

- Notice that this is exactly the eigenvector/eigenvalue equation, so $(\lambda, v)$ is an eigenpair of $A$.
- Thus, for each eigenpair $(\lambda, v)$, $e^{\lambda t} v$ is a linearly independent solution. If we had $n$ eigenpairs, we could get $n$ solutions.

# Using an eigenbasis for general solutions

- Let $\dot{z} = Az$, where $A$ is an $n \times n$ matrix and $z(t)$ is a length-n vector.
- Then if $(\lambda_1, v_1), (\lambda_2, v_2), \ldots, (\lambda_n, v_n)$ is an eigenbasis of $A$, then

$$z(t) = c_1 v_1 e^{\lambda_1 t} + \cdots + c_n v_n e^{\lambda_n t}$$

is the general solution for $z(t)$.

$\dot{z} = Az$

Ex $A = \begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix}$ Eigenvalues: 1, 2

$\lambda_1 = 1$ $\begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \left.\begin{matrix} x = x \\ -4x + 2y = y \end{matrix}\right\} \Rightarrow \begin{matrix} -4x = -y \\ y = 4x \end{matrix}$ $v_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

subtract x

$\lambda_2 = 2$ $\begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \Rightarrow \left.\begin{matrix} x = 2x \\ -4x + 2y = 2y \end{matrix}\right\} \Rightarrow \begin{matrix} x = 0 \\ y = y \end{matrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$z = c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} c_1 e^{t} \\ 4c_1 e^{t} + c_2 e^{2t} \end{bmatrix}$$

Ex. $c_1 = 1$, $c_2 = -2$

$$\begin{bmatrix} e^{t} \\ 4e^{t} - 2e^{2t} \end{bmatrix}$$

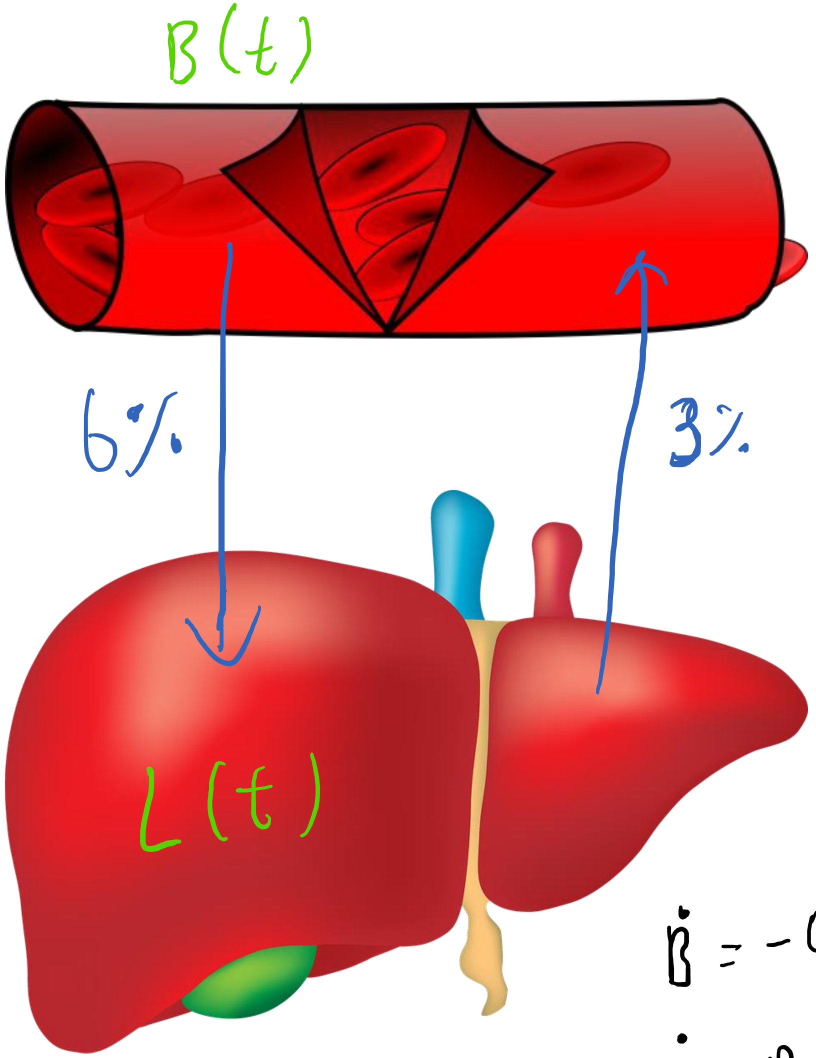
$$c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t}$$

$$\begin{bmatrix} c_1 e^{t} \\ 4c_1 e^{t} \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} c_1 e^{t} \\ 4c_1 e^{t} + c_2 e^{2t} \end{bmatrix}$$

# Application (Bittinger, 9.3, Ex. 5)

- 3mg of glactosyl human serum albumin (Tc-GSA) is injected into a patient's bloodstream to measure liver function. After injection, Tc-GSA is transferred from the blood to the liver at 6% per minute, and from the liver into blood at 3% per minute.
- How much Tc-GSA is in the blood or liver as a function of time?




$B(0) = 3mg$

$L(0) = 0mg$

$$\begin{array}{c} \boxed{B} \\ 0.06B \downarrow \quad \uparrow 0.03L \\ \boxed{L} \end{array}$$

$\dot{B} = -0.06B + 0.03L$

$\dot{L} = 0.06B - 0.03L$

# Application (continued)

$$0.03L = \dot{B} + 0.06B$$
$$3L = 100\dot{B} + \cancel{6}B$$
$$L = \frac{100}{3}\dot{B} + 2B$$

- $$\begin{cases}\dot{B} = -0.06B + 0.03L \\ \dot{L} = \ \ 0.06B - 0.03L\end{cases}$$

get 2nd derivative

$$\ddot{B} = -0.06\dot{B} + 0.03\dot{L}$$

$$\ddot{B} = -0.06\dot{B} + 0.03\left[0.06B - 0.03L\right]$$

get rid of $\dot{L}$

$$\ddot{B} = -0.06\dot{B} + 0.0018B - 0.0009L$$

$$\ddot{B} = -0.06\dot{B} + 0.0018B - 0.0009\left[\frac{100}{3}\dot{B} + 2B\right]$$

$$\ddot{B} = -0.06\dot{B} + \underline{0.0018B} - 0.03\dot{B} \underline{- 0.0018B}$$

$$\ddot{B} = -0.09\dot{B}$$

$$\ddot{B} + 0.09\dot{B} = 0$$

# Application (continued)

- $\ddot{B} + 0.09\dot{B} = 0$

$\lambda^2 + 0.09\lambda = 0$

$\lambda(\lambda + 0.09) = 0$

$\lambda = 0, \; -0.09$

$B = c_1 e^{0t} + c_2 e^{-0.09t}$

$B = c_1 + c_2 e^{-0.09t}$

$L = \frac{100}{3}\dot{B} + 2B$

$\dot{B} = -0.09 c_2 e^{-0.09t}$

$L = -3c_2 e^{-0.09t} + 2c_1 + 2c_2 e^{-0.09t}$

$L = 2c_1 - c_2 e^{-0.09t}$

# Initial Value Problem

- $\begin{cases} B(t) = c_1 + c_2 e^{-0.09t} \\ L(t) = 2c_1 - c_2 e^{-0.09t} \end{cases}$, with initial values $\begin{cases} B(0) = 3 \\ L(0) = 0 \end{cases}$

$$3 = B(0) = c_1 + c_2 e^{-0.09 \cdot 0} = c_1 + c_2$$

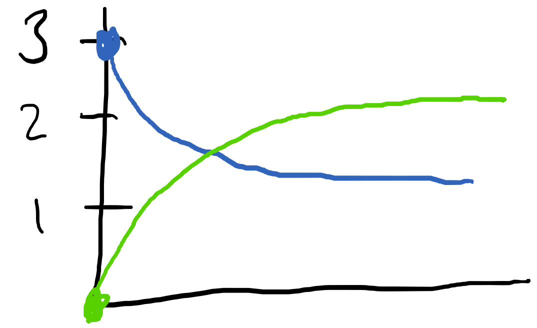
$$0 = L(0) = 2c_1 - c_2 e^{-0.09 \cdot 0} = 2c_1 - c_2$$

$$3 = 3c_1$$

$$c_1 = 1 \Rightarrow c_2 = 2$$

$$\underline{B(t) = 1 + 2e^{-0.09t}} \qquad \underline{L(t) = 2 - 2e^{-0.09t}}$$

$$\lim_{t \to \infty} B(t) = 1 \text{ mg} \qquad \lim_{t \to \infty} L(t) = 2 \text{ mg.}$$



# Solving as a matrix equation

- $\begin{cases} \dot{B} = -0.06B + 0.03L \\ \dot{L} = \ \ 0.06B - 0.03L \end{cases}$ converts to $\begin{bmatrix} \dot{B} \\ \dot{L} \end{bmatrix} = \begin{bmatrix} -0.06 & 0.03 \\ 0.06 & -0.03 \end{bmatrix} \begin{bmatrix} B \\ L \end{bmatrix}$

Eigendecomposition $\det(\lambda I - A) = 0$

$$0 = \begin{vmatrix} \lambda + 0.06 & -0.03 \\ -0.06 & \lambda + 0.03 \end{vmatrix} = \lambda^2 + 0.09\lambda + 0.0018 - 0.0018 = 0$$

$$\lambda^2 + 0.09\lambda = 0$$

$$\lambda(\lambda + 0.09) = 0 \Rightarrow \lambda = 0, -0.09$$

$$\lambda_1 = 0 \quad \begin{bmatrix} -0.06 & 0.03 \\ 0.06 & -0.03 \end{bmatrix} \begin{bmatrix} B \\ L \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -0.06B + 0.03L = 0 \quad v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$L = 2B$$

$$\lambda_2 = -0.09 \quad \begin{bmatrix} -0.06 & 0.03 \\ 0.06 & -0.03 \end{bmatrix} \begin{bmatrix} B \\ L \end{bmatrix} = \begin{bmatrix} -0.09B \\ -0.09L \end{bmatrix} \Rightarrow -0.06B + 0.03L = -0.09B$$

$$0.03L = -0.03B$$

$$L = -B \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

# Using eigenvectors for general solution

- $\lambda_1 = 0, v_1 = \begin{bmatrix}1\\2\end{bmatrix}$ and $\lambda_2 = -0.09, v_2 = \begin{bmatrix}1\\-1\end{bmatrix}$

$$\begin{bmatrix}B\\L\end{bmatrix} = c_1 e^{0t}\begin{bmatrix}1\\2\end{bmatrix} + c_2 e^{-0.09t}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\begin{bmatrix}B\\L\end{bmatrix} = c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2 e^{-0.09t}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}c_1 + c_2 e^{-0.09t}\\ 2c_1 - c_2 e^{-0.09t}\end{bmatrix}$$

Initial value problem can then be solved by plugging in $B(0)$, $L(0)$.

# Try it out

- $\begin{cases} \dot{x} = y \\ \dot{y} = 3x + 2y \end{cases}$ and $x(0) = 0, y(0) = 4$
- Step 1. Convert to $z = Az$, where $z = \begin{bmatrix} x \\ y \end{bmatrix}$.

A: $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$
B: $A = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$
C: $A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$
D: $A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$
E: None of the above

- Step 2. Find the eigenvalues.

$A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$ $\quad |A - \lambda I| = 0$

or $|\lambda I - A| = 0$

$\begin{vmatrix} \lambda & -1 \\ -3 & \lambda - 2 \end{vmatrix} = \lambda^2 - 2\lambda - 3 = 0$

$(\lambda - 3)(\lambda + 1) = 0$

$\lambda = -1, 3$

A: $\lambda = -1, 1$
B: $\lambda = 1, 2$
C: $\lambda = 1, 3$
D: $\lambda = -1, 3$
E: None of the above

# Try it out (continued)

A: $v_1 = \begin{bmatrix}1\\1\end{bmatrix}, v_2 = \begin{bmatrix}1\\3\end{bmatrix}$

B: $v_1 = \begin{bmatrix}1\\-1\end{bmatrix}, v_2 = \begin{bmatrix}1\\3\end{bmatrix}$

C: $v_1 = \begin{bmatrix}1\\1\end{bmatrix}, v_2 = \begin{bmatrix}1\\-3\end{bmatrix}$

D: $v_1 = \begin{bmatrix}1\\-1\end{bmatrix}, v_2 = \begin{bmatrix}1\\-3\end{bmatrix}$

E: None of the above

- Find the eigenvectors.

$$\begin{bmatrix}0 & 1\\3 & 2\end{bmatrix} \quad \lambda_1 = -1, \quad \lambda_2 = 3$$

$$\lambda_1 = -1 \quad \begin{bmatrix}0 & 1\\3 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-x\\-y\end{bmatrix} \quad y = -x \quad v_1 = \begin{bmatrix}1\\-1\end{bmatrix}$$

$$\lambda_2 = 3 \quad \begin{bmatrix}0 & 1\\3 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3x\\3y\end{bmatrix} \quad y = 3x \quad v_2 = \begin{bmatrix}1\\3\end{bmatrix}$$

- What's the general solution?

A: $\begin{bmatrix}x\\y\end{bmatrix} = c_1 e^{-t}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{3t}\begin{bmatrix}1\\3\end{bmatrix}$

B: $\begin{bmatrix}x\\y\end{bmatrix} = c_1 e^{-t}\begin{bmatrix}1\\-1\end{bmatrix} + c_2 e^{3t}\begin{bmatrix}1\\3\end{bmatrix}$

C: $\begin{bmatrix}x\\y\end{bmatrix} = c_1 e^{-t}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{3t}\begin{bmatrix}1\\-3\end{bmatrix}$

D: $\begin{bmatrix}x\\y\end{bmatrix} = c_1 e^{-t}\begin{bmatrix}1\\-1\end{bmatrix} + c_2 e^{3t}\begin{bmatrix}1\\-3\end{bmatrix}$

E: None of the above

# Try it out (cont., Initial value problem)

$t$

- $x(0) = 0, y(0) = 4$

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 4 \end{bmatrix} = c_1 e^{0}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{0}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 4 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ -c_1 + 3c_2 \end{bmatrix}$$

$$\Rightarrow \begin{array}{rl} & c_1 + c_2 = 0 \\ +) & -c_1 + 3c_2 = 4 \\ \hline & 4c_2 = 4 \end{array} \Rightarrow \begin{array}{l} c_2 = 1 \\ c_1 = -1 \end{array}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = -e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{3t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

A: $c_1 = -1, c_2 = -1$
B: $c_1 = 1, c_2 = -1$
C: $c_1 = -1, c_2 = 1$
D: $c_1 = 1, c_2 = 1$
E: None of the above