Matrix ODE representations
Lecture 10b: 2021-07-28

MAT A35 – Summer 2021 – UTSC

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Homogeneous system as matrix equation

- We can rewrite a homogeneous system of 1st-order ODEs as a matrix-vector ODE.
- $\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases} \quad \rightarrow \quad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
- The matrix-form equation $\dot{z} = Az$ has solutions $z(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

Solution to a matrix-form equation

- If $y' = ky$, then $y = c_0 e^{kx}$ for single-variable ODEs.
- If $\dot{z} = Az$, then perhaps $z = e^{\lambda t} v$, for some constant λ and vector v.

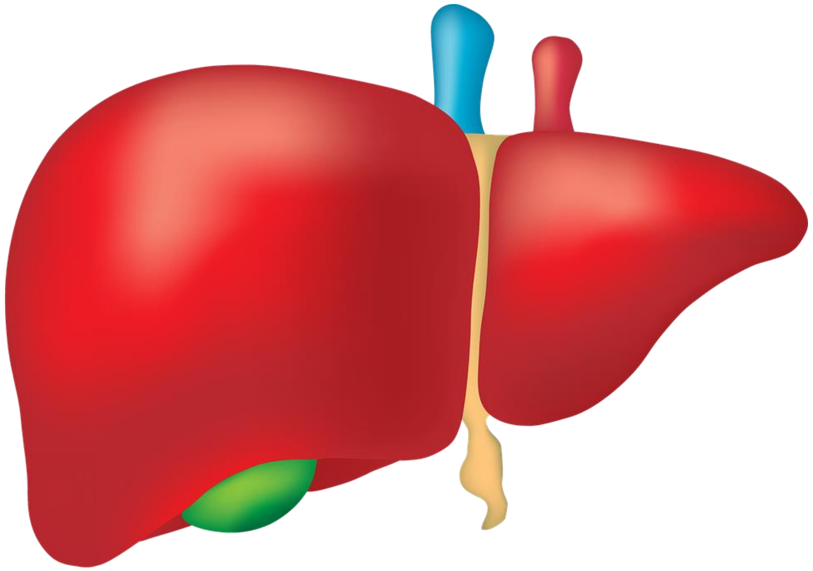
- Notice that this is exactly the eigenvector/eigenvalue equation, so (λ, v) is an eigenpair of A.
- Thus, for each eigenpair (λ, v), $e^{\lambda t} v$ is a linearly independent solution. If we had n eigenpairs, we could get n solutions.

Using an eigenbasis for general solutions

- Let $\dot{z} = Az$, where A is an $n \times n$ matrix and $z(t)$ is a length-n vector.
- Then if $(\lambda_1, v_1), (\lambda_2, v_2), \ldots, (\lambda_n, v_n)$ is an eigenbasis of A, then
$$z(t) = c_1 v_1 e^{\lambda_1 t} + \cdots c_n v_n e^{\lambda_n t}$$
is the general solution for $z(t)$.

Application (Bittinger, 9.3, Ex. 5)

- 3mg of glactosyl human serum albumin (Tc-GSA) is injected into a patient's bloodstream to measure liver function. After injection, Tc-GSA is transferred from the blood to the liver at 6% per minute, and from the liver into blood at 3% per minute.
- How much Tc-GSA is in the blood or liver as a function of time?



Application (continued)

- $\begin{cases} \dot{B} = -0.06B + 0.03L \\ \dot{L} = \ \ 0.06B - 0.03L \end{cases}$

Application (continued)

- $\ddot{B} + 0.09\dot{B} = 0$

$$L = \frac{100}{3}\dot{B} + 2B$$

Initial Value Problem

- $\begin{cases} B(t) = c_1 + c_2 e^{-0.09t} \\ L(t) = 2c_1 - c_2 e^{-0.09t} \end{cases}$, with initial values $\begin{cases} B(0) = 3 \\ L(0) = 0 \end{cases}$

Solving as a matrix equation

- $\begin{cases} \dot{B} = -0.06B + 0.03L \\ \dot{L} = \ \ 0.06B - 0.03L \end{cases}$ converts to $\begin{bmatrix} \dot{B} \\ \dot{L} \end{bmatrix} = \begin{bmatrix} -0.06 & 0.03 \\ 0.06 & -0.03 \end{bmatrix} \begin{bmatrix} B \\ L \end{bmatrix}$

Using eigenvectors for general solution

- $\lambda_1 = 0, v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\lambda_2 = -0.09, v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Try it out

- $\begin{cases} \dot{x} = y \\ \dot{y} = 3x + 2y \end{cases}$ and $x(0) = 0, y(0) = 4$
- Step 1. Convert to $z = Az$, where $z = \begin{bmatrix} x \\ y \end{bmatrix}$.

A: $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$
B: $A = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$
C: $A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$
D: $A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$
E: None of the above

- Step 2. Find the eigenvalues.

A: $\lambda = -1, 1$
B: $\lambda = 1, 2$
C: $\lambda = 1, 3$
D: $\lambda = -1, 3$
E: None of the above

Try it out (continued)

- Find the eigenvectors.

A: $v_1 = \begin{bmatrix}1\\1\end{bmatrix}, v_2 = \begin{bmatrix}1\\3\end{bmatrix}$

B: $v_1 = \begin{bmatrix}1\\-1\end{bmatrix}, v_2 = \begin{bmatrix}1\\3\end{bmatrix}$

C: $v_1 = \begin{bmatrix}1\\1\end{bmatrix}, v_2 = \begin{bmatrix}1\\-3\end{bmatrix}$

D: $v_1 = \begin{bmatrix}1\\-1\end{bmatrix}, v_2 = \begin{bmatrix}1\\-3\end{bmatrix}$

E: None of the above

- What's the general solution?

A: $\begin{bmatrix}x\\y\end{bmatrix} = c_1 e^{-t}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{3t}\begin{bmatrix}1\\3\end{bmatrix}$

B: $\begin{bmatrix}x\\y\end{bmatrix} = c_1 e^{-t}\begin{bmatrix}1\\-1\end{bmatrix}, +c_2 e^{3t}\begin{bmatrix}1\\3\end{bmatrix}$

C: $\begin{bmatrix}x\\y\end{bmatrix} = c_1 e^{-t}\begin{bmatrix}1\\1\end{bmatrix}, +c_2 e^{3t}\begin{bmatrix}1\\-3\end{bmatrix}$

D: $\begin{bmatrix}x\\y\end{bmatrix} = c_1 e^{-t}\begin{bmatrix}1\\-1\end{bmatrix}, +c_2 e^{3t}\begin{bmatrix}1\\-3\end{bmatrix}$

E: None of the above

Try it out (cont., Initial value problem)

- $x(0) = 0, y(0) = 4$

A: $c_1 = -1, c_2 = -1$
B: $c_1 = 1, c_2 = -1$
C: $c_1 = -1, c_2 = 1$
D: $c_1 = 1, c_2 = 1$
E: None of the above