Phase Portraits
Lecture 10c: 2021-07-28

MAT A35 – Summer 2021 – UTSC

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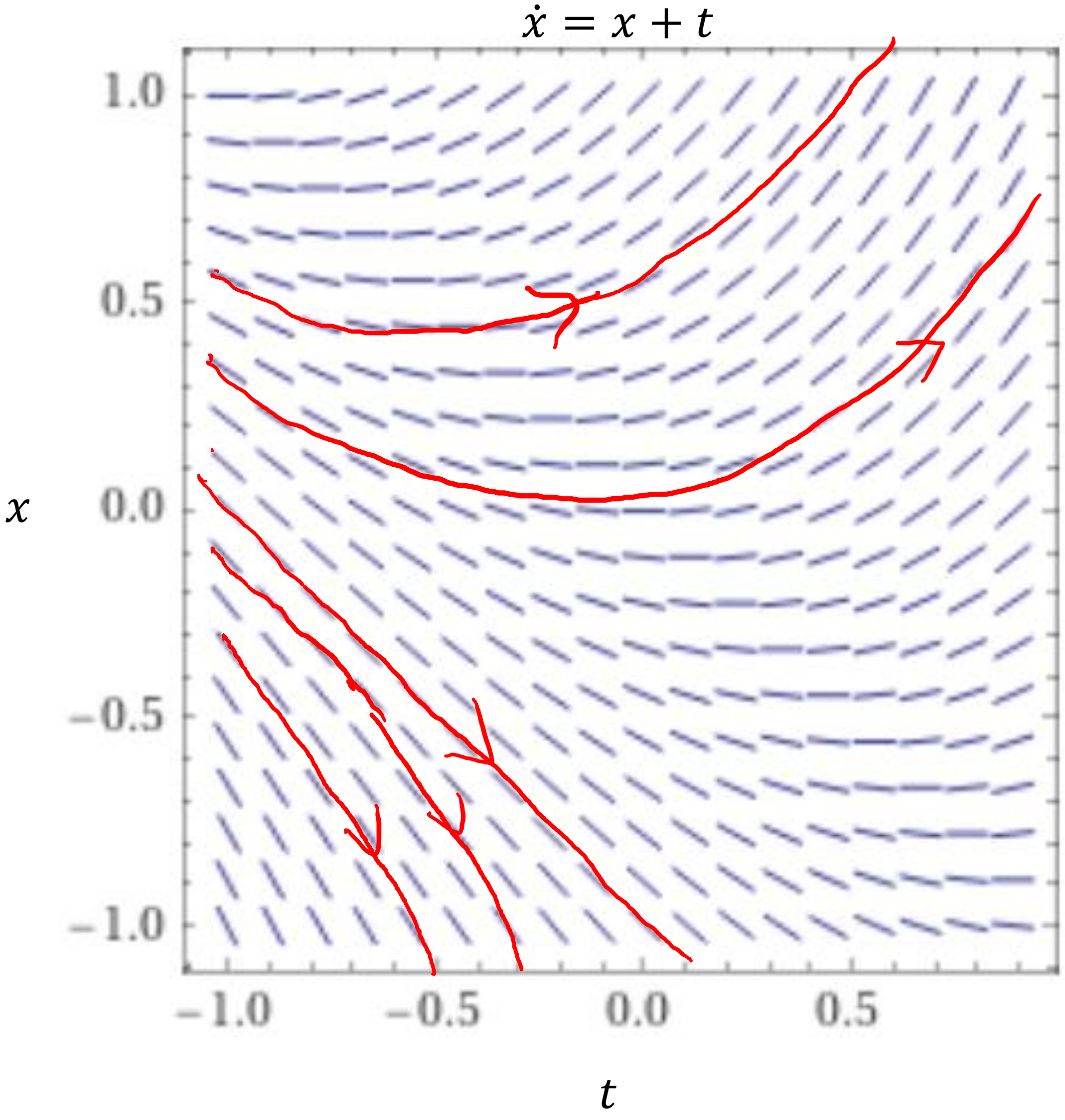
Recall: direction field for a 1st-order ODE

- A direction field graphs out the slopes of all solutions going through a point.
- We can visualize different solutions by drawing trajectory curves that are always tangent to the direction field.



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Recall: phase lines for a 1st-order ODE

- Notice that if $\dot{x} = f(x)$, then the slope has no t-dependence.
- We can draw just the x-axis as a phase line.

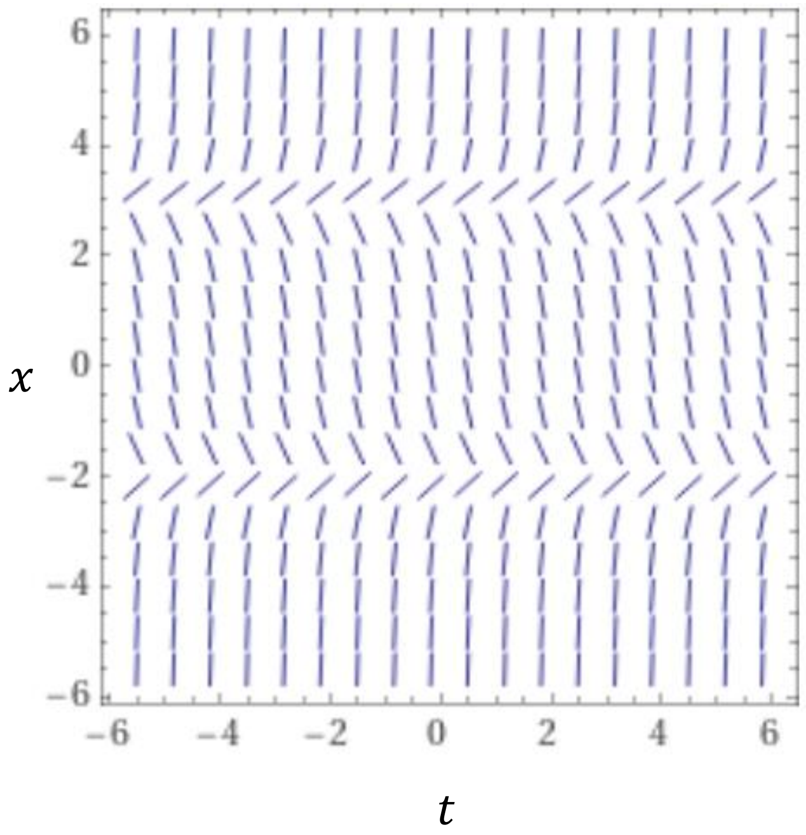
$\dot{x} = 0 = x^2 - x - 6$

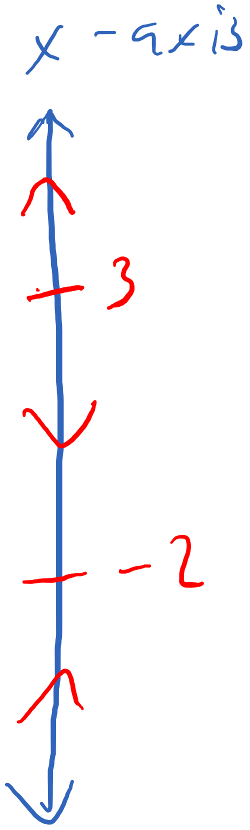
$0 = (x-3)(x+2)$

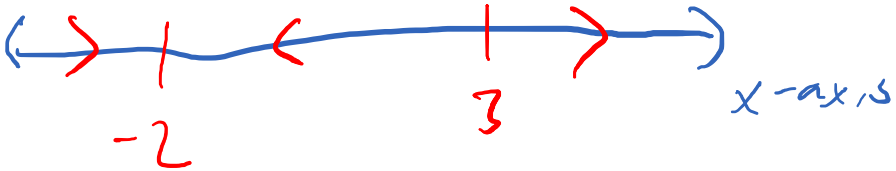
$\Rightarrow x = -2, 3$

are equilibria

$\dot{x} = x^2 - x - 6$







System of two 1st-order ODEs

- $\begin{cases} \dot{x} = x + y - \sin t \\ \dot{y} = x^2 + y^2 - \ln t \end{cases}$ (nonautonomous)
- How many dimensions do nonautonomous systems need to draw direction fields?
- $\begin{cases} \dot{x} = x + y \\ \dot{y} = x^2 + y^2 \end{cases}$ (autonomous system)
- How many dimensions do autonomous systems need to draw direction fields?
- How many dimensions do autonomous systems need to draw phase "lines"?

3 dims for x, y, t

A: 1
B: 2
C: 3
D: 4
E: None of the above

could draw using 3 dim.

2 dims for just x & y

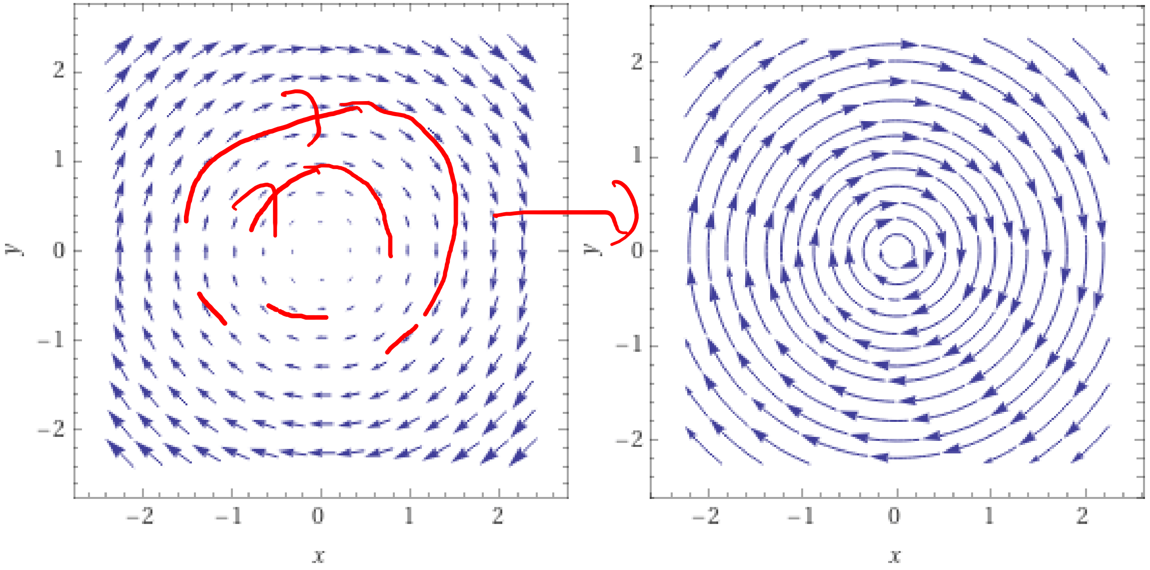
Plotting vector fields and trajectories

- Consider $\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$
- The system associates a direction and a magnitude for every point in $\mathbb{R}^2$, telling you what direction trajectories go.
- WolframAlpha: "vector field {f(x,y), g(x,y)}"
- Ex: "vector field {y, -x}" or "integral curves {y, -x}"
- Specify limits by adding "x=-3..3, y=-3..3" after

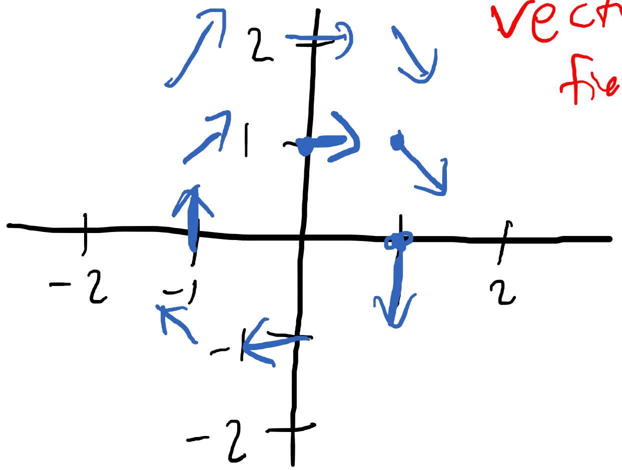
Ex. $\dot{x} = y$
$\dot{y} = -x$

x	y	$\dot{x}$	$\dot{y}$
0	1	1	0
1	1	1	-1
1	0	0	-1
-1	0	0	1

vector field trajectory



vector field

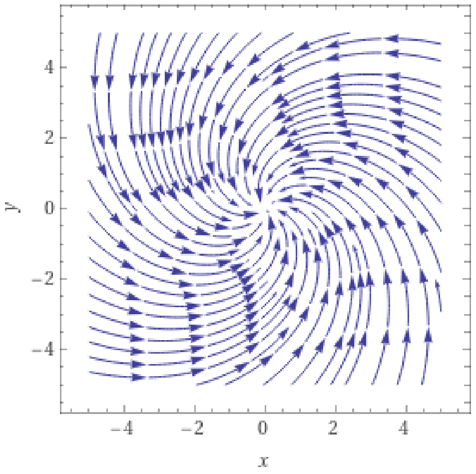


Try it out

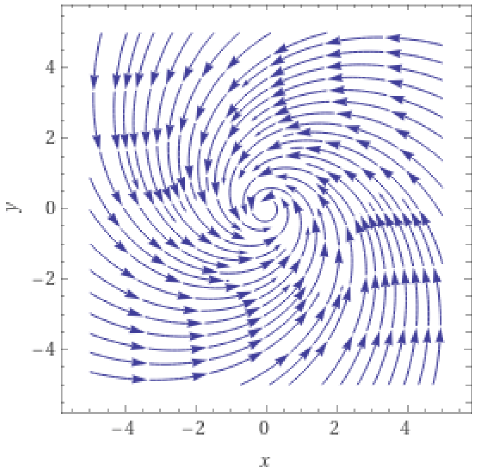
- Which of the following is the integral curves for the system, plotted for x and y both between -5 and 5?

$$\dot{x} = 4x - y - \left(x + \frac{3}{2}y\right)(x^2 + y^2)$$
$$\dot{y} = x + 4y + \left(\frac{3}{2}x - y\right)(x^2 + y^2)$$

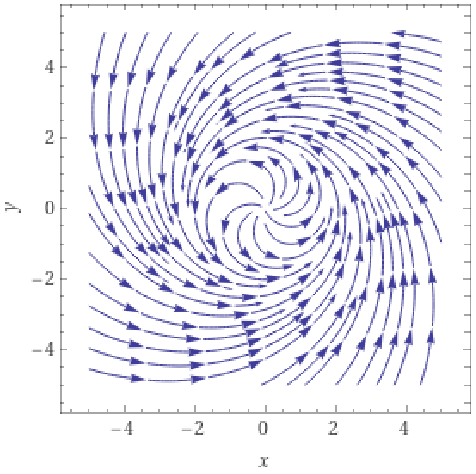
A:



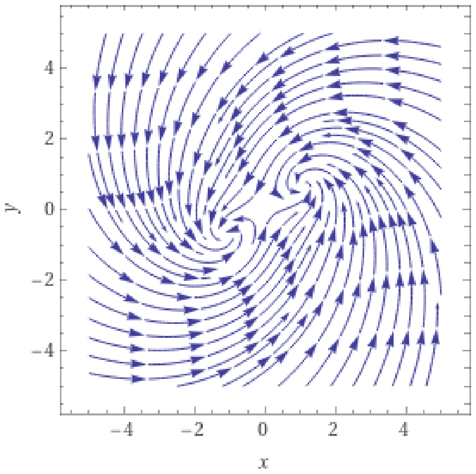
B:



C:



D:



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Phase plane analysis

- Consider the autonomous homogeneous 2D linear system with constant coefficients

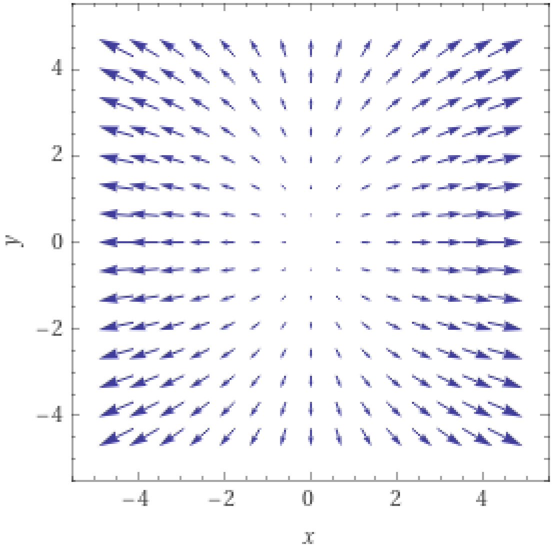
$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}, A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

- Also, notice that the origin is always an equilibrium for a linear system.

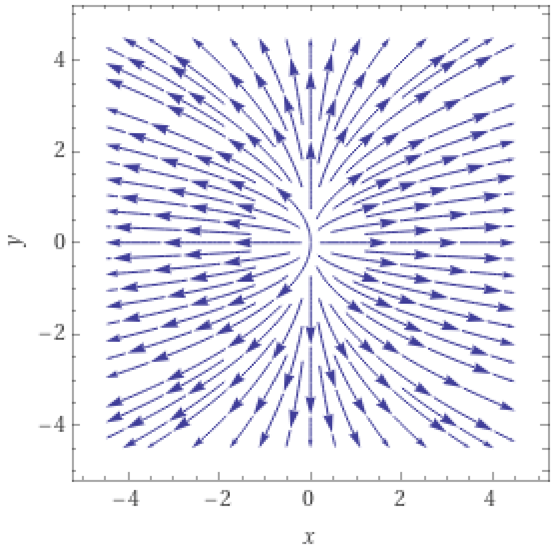
Ex $\quad A = \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$

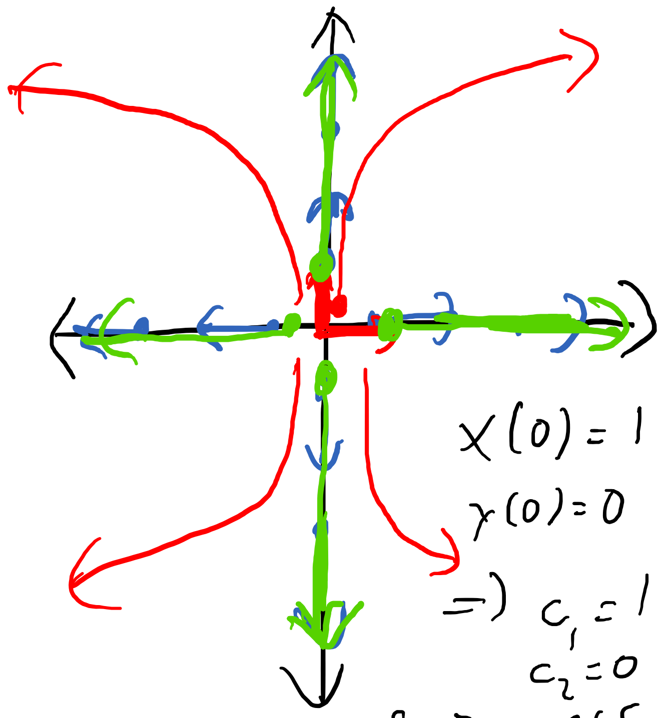
$$\begin{bmatrix}x\\ y\end{bmatrix} = c_1 e^{2t}\begin{bmatrix}1\\ 0\end{bmatrix} + c_2 e^{t}\begin{bmatrix}0\\ 1\end{bmatrix}$$

vector field



trajectories





$x(0) = 1$

$y(0) = 0$

$\Rightarrow \; c_1 = 1$

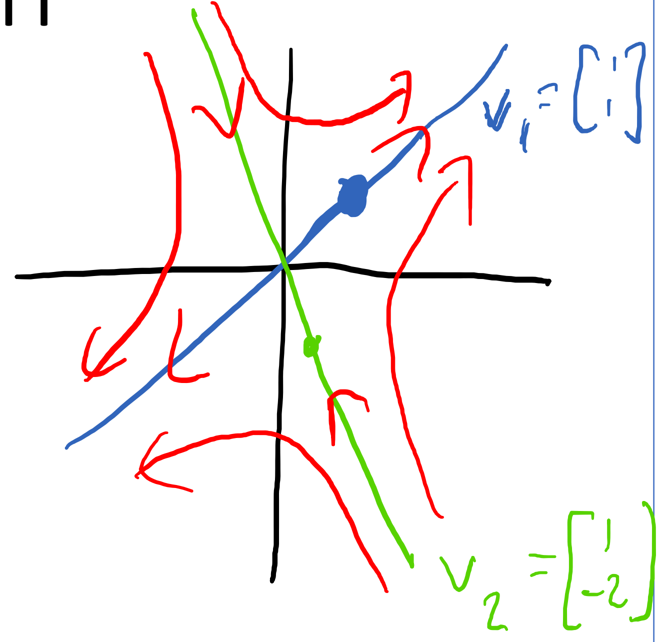
$c_2 = 0$

$\begin{bmatrix}x\\ y\end{bmatrix} = e^{2t}\begin{bmatrix}1\\ 0\end{bmatrix}$

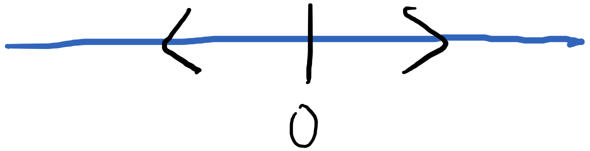
Using the eigendecomposition

- If (λ_1, v_1) and (λ_2, v_2) is an eigendecomposition of A, then the general solution describing any trajectory is

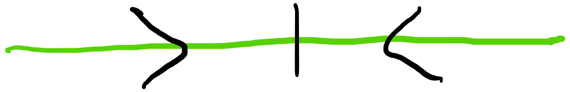
$$c_1 e^{\lambda_1 t} \boxed{v_1} + c_2 e^{\lambda_2 t} v_2$$



- We can qualitatively analyze the behavior of the system by looking at the eigenvalues and eigenvectors.
- Consider $f(t) = ce^{\lambda t}$.
- If $\lambda > 0$, we get exponential growth away from 0.



- If $\lambda < 0$, we get exponential decay towards 0.



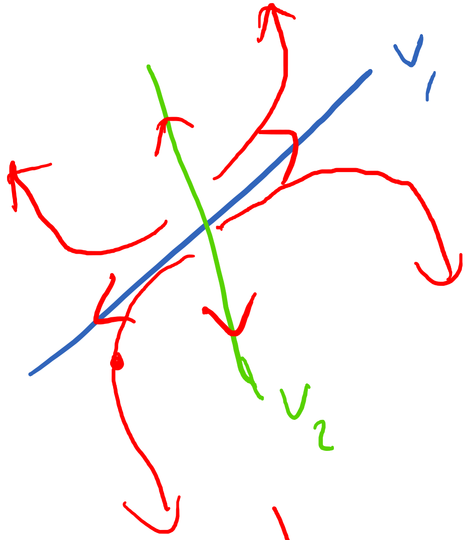
Sign and stability

- If eigenvalues are positive (or have a positive real part), then trajectories go away from the origin. (unstable node)
- If eigenvalues are negative (or have a negative real part), then trajectories go towards the origin. (asymptotically stable node)
- If eigenvalues have opposite signs, then we have a saddle point, as trajectories come in along one eigenvector, and leave along the other. (unstable, saddle point)

Ex. $\lambda = 2, 5$

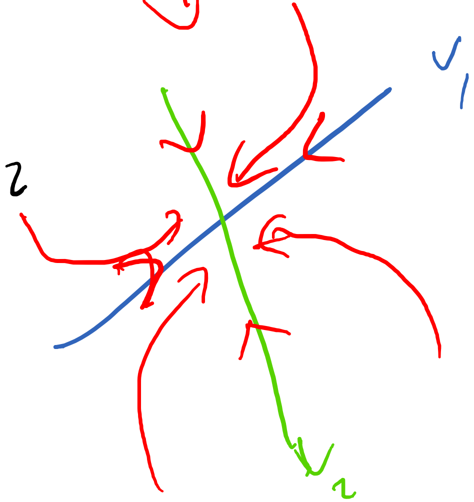
$$z = c_1 e^{2t} v_1 + c_2 e^{5t} v_2$$

As $t \to \infty$, both get bigger



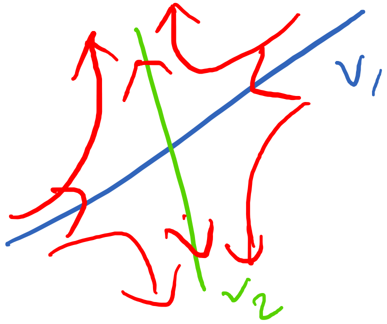
Ex. $\lambda = -2, -5$

$$z = c_1 e^{-2t} v_1 + c_2 e^{-5t} v_2$$



Ex. $\lambda = -2, 5$

$$z = c_1 e^{-2t} v_1 + c_2 e^{5t} v_2$$



Complex eigenvalues

- Recall complex eigenvalues come in pairs $\lambda_{1,2} = a \pm bi$.
- Solutions look like

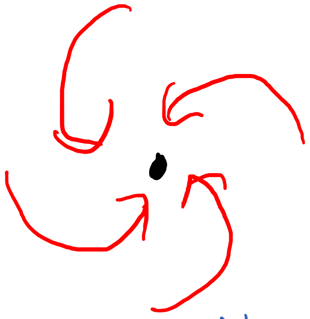
rotation

$$z = c_1 v_1 e^{at} \cos bt + c_2 v_2 e^{at} \sin bt$$

stability

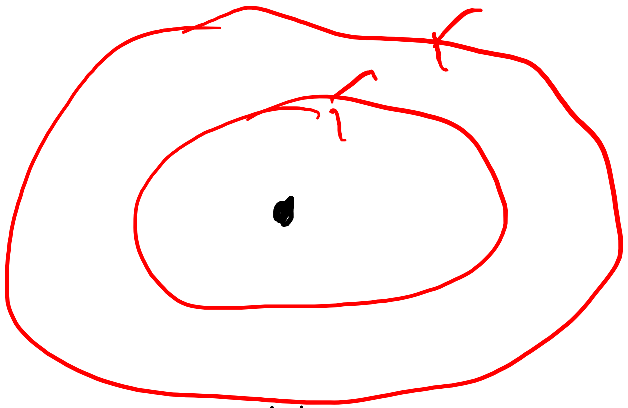
- The sign of the real part a determines if the trajectories go inward (stable) or outward (unstable).
- The imaginary term means that the trajectories have a rotational component; i.e. might spiral in or out, or form a circle.

If $a < 0$



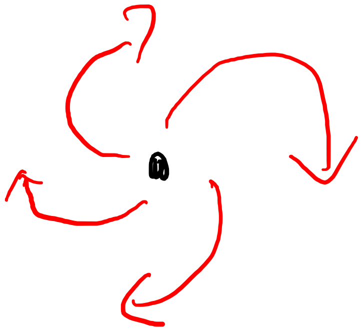
asymp. stable spiral

If $a = 0$



"stable" center

If $a > 0$

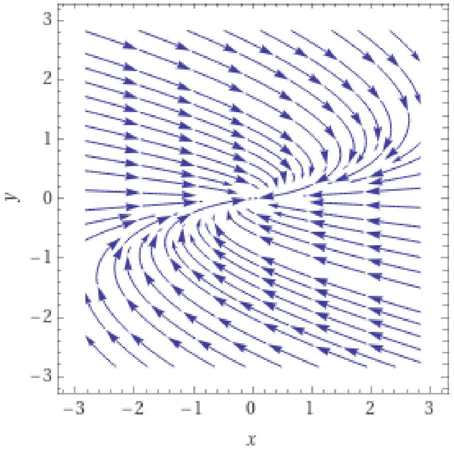


unstable spiral

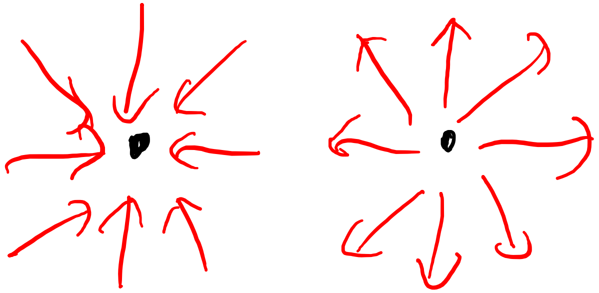
Degenerate special cases

$A = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$

- Sometimes, if $\lambda_1 = \lambda_2$, there is only one eigenvector. Then we have an *improper* node that's hard to draw.
  - Sign still determines stable vs unstable.



- If $\lambda_1 = \lambda_2$ and we have two eigenvectors, then we have a *proper* node, which looks like a star.



Note: $A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ for proper nodes

(diagonal)

Summarizing everything

- $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
- The origin (0,0) is always an equilibrium point.
- We can understand the behavior around the origin by looking at the eigenvalues of A.
- Positive real parts mean that the trajectories go outward.
- Negative real parts mean that the trajectories go inward.
- Opposite sign eigenvalues mean you have a saddle point.
- Nonzero imaginary components mean that trajectories spiral.

Try it out

- $\lambda_1 = 4, \lambda_2 = -2$
- $\lambda_1 = -3, \lambda_2 = -1$
- $\lambda_1 = 2, \lambda_2 = 3$
- $\lambda_1 = 3, \lambda_2 = 3$
- $\lambda_1 = 3 + 2i, \lambda_2 = 3 - 2i$
- $\lambda_{1,2} = -1 \pm 2i$
- $\lambda_{1,2} = \pm 4i$

A: Asymptotically Stable
B: Stable
C: Unstable
D: ???
E: None of the above

A: Node (incl. (im)proper)
B: Saddle Point
C: Spiral
D: Center
E: None of the above

Special note: weird stuff can happen when $\lambda = 0$, which we won't deal with.

Example

- Classify the behavior around the origin of $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Example

- Classify the behavior around the origin of $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Try it out

- Classify the behavior around the origin of $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

A: Asymptotically Stable
B: Stable
C: Unstable
D: ???
E: None of the above

A: Node (incl. (im)proper)
B: Saddle Point
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D: Center
E: None of the above