Nonlinear Phase Portraits

Lecture ~~10d~~ 11a: 2021-07-28

MAT A35 – Summer 2021 – UTSC

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Summarizing everything

- $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- The origin (0,0) is always an equilibrium point, and the eigenvalues of A determine the behavior at that equilibrium.
- Sign:
  - Positive real parts → trajectories go outward (unstable).
  - Negative real parts → trajectories go inward (asymptotically stable).
  - Opposite sign eigenvalues → saddle point (unstable).
- Real vs complex:
  - Real eigenvalues give either nodes (if both same sign) or saddle points.
  - Nonzero imaginary components mean that trajectories spiral.
  - Pure imaginary eigenvalues give rise to a “center”. (stable)

Try it out

- $\lambda_1 = 4, \lambda_2 = -2$
- $\lambda_1 = -3, \lambda_2 = -1$
- $\lambda_1 = 2, \lambda_2 = 3$
- $\lambda_1 = 3, \lambda_2 = 3$
- $\lambda_1 = 3 + 2i, \lambda_2 = 3 - 2i$
- $\lambda_{1,2} = -1 \pm 2i$
- $\lambda_{1,2} = \pm 4i$

B saddle, different signs

node, but it might be proper or improper depending on eigenvectors

A: Node (incl. (im)proper)
B: Saddle Point
C: Spiral
D: Center
E: None of the above

Special note: weird stuff can happen when $\lambda = 0$, which we won't deal with.

Example

- Classify the behavior around the origin of $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Eigenvalues:

$$\begin{vmatrix} \lambda - 1 & -3 \\ -3 & \lambda - 1 \end{vmatrix} = \lambda^2 - 2\lambda + 1 - 9 = 0$$

$$\lambda^2 - 2\lambda - 8 = 0 \cancel{4}$$

$$\Rightarrow (\lambda - 4)(\lambda + 2) = 0$$

$$\lambda = -2, 4$$

<u>Saddle</u> <u>pt</u>

Example

- Classify the behavior around the origin of $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Eigenvalues

$$\begin{vmatrix} \lambda-1 & -3 \\ 0 & \lambda-1 \end{vmatrix} = (\lambda-1)^2 = 0$$

$\lambda = 1$, mult 2.

unstable node

To determine if (im)proper: check # eigenvectors

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$\Rightarrow x + 3y = x$

$\Rightarrow y = 0$

$\Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Only 1 eigenvector, so improper node.

Try it out

- Classify the behavior around the origin of $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Eig.

$$\begin{vmatrix} \lambda-1 & 3 \\ -3 & \lambda-1 \end{vmatrix} = 0$$

$$(\lambda-1)^2 + 9 = 0$$

$$(\lambda-1)^2 = -9$$

$$\lambda - 1 = \pm 3i$$

$$\lambda = 1 \pm 3i$$

A: Asymptotically Stable
B: Stable
C: Unstable
D: ???
E: None of the above

A: Node (incl. (im)proper)
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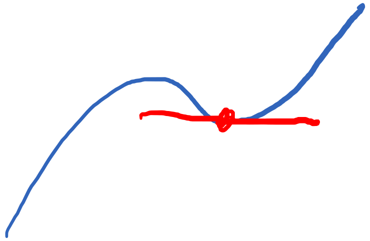
Nonlinear autonomous systems and Jacobians



- $\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$
- Equilibrium points when $\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases}$
- We can approximate a function around a point using its derivative at that point.
- The *Jacobian* of the system is the analogue of the derivative:

$$J(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

Ex

$$\begin{cases} \dot{x} = x^2 + y^2 - 1 \\ \dot{y} = 2xy \end{cases}$$

$0 = x^2 + y^2 - 1$

$0 = 2xy$

$\Rightarrow x = 0$ or $y = 0$

If $x = 0$, $y = \pm 1$

If $y = 0$, $x = \pm 1$

Equilibria: $(0,1)$ $(0,-1)$

$(1,0)$ $(-1,0)$

$$J(x,y) = \begin{bmatrix} 2x & 2y \\ 2y & 2x \end{bmatrix}$$

Nonlinear equilibria behavior

$$J(x,y) = \begin{bmatrix} 2x & 2y \\ 2y & 2x \end{bmatrix}$$

- Around each equilibrium, we can approximate its behavior by looking at the Jacobian matrix' eigenvalues.

$$J(0,1) = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \qquad \begin{vmatrix} \lambda & -2 \\ -2 & \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 4 = 0, \quad \lambda = \pm 2$$

saddle pt.

$\lambda_1 = 2$ $\quad \lambda_2 = -2$

$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$J(0,-1) = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} \qquad \begin{vmatrix} \lambda & 2 \\ 2 & \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 4 = 0, \quad \lambda = \pm 2$$

saddle pt.

$\lambda_1 = 2$ $\quad v_2 = -2$

$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$J(1,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad \lambda = 2, \text{ mult } 2.$$

$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

unstable proper node

$$J(-1,0) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \qquad \lambda = -2, \text{ mult } 2$$

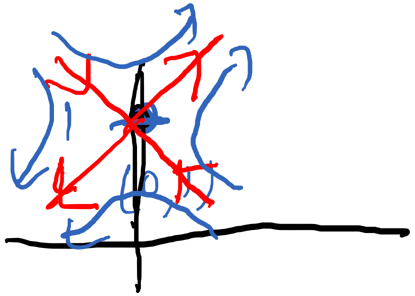
stable proper node

Local phase portraits

$(0, 1)$

$\lambda_1 = 2 \quad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

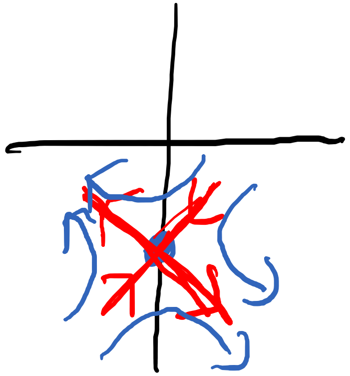
$\lambda_2 = -2 \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$



$(0, -1)$

$\lambda_1 = 2 \quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

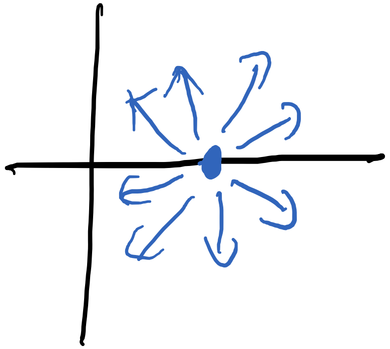
$\lambda_2 = -2 \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



$(1, 0)$

$\lambda = 2$, mult 2

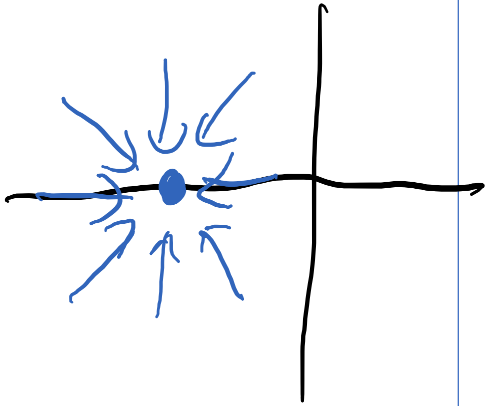
proper node



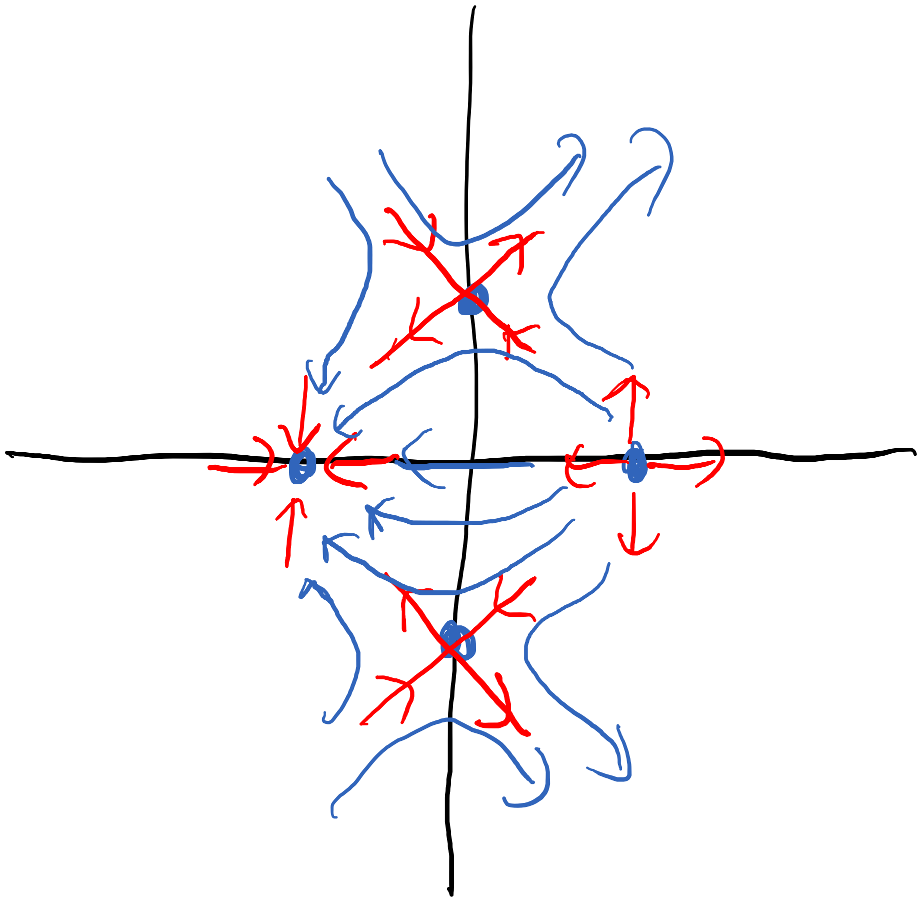
$(-1, 0)$

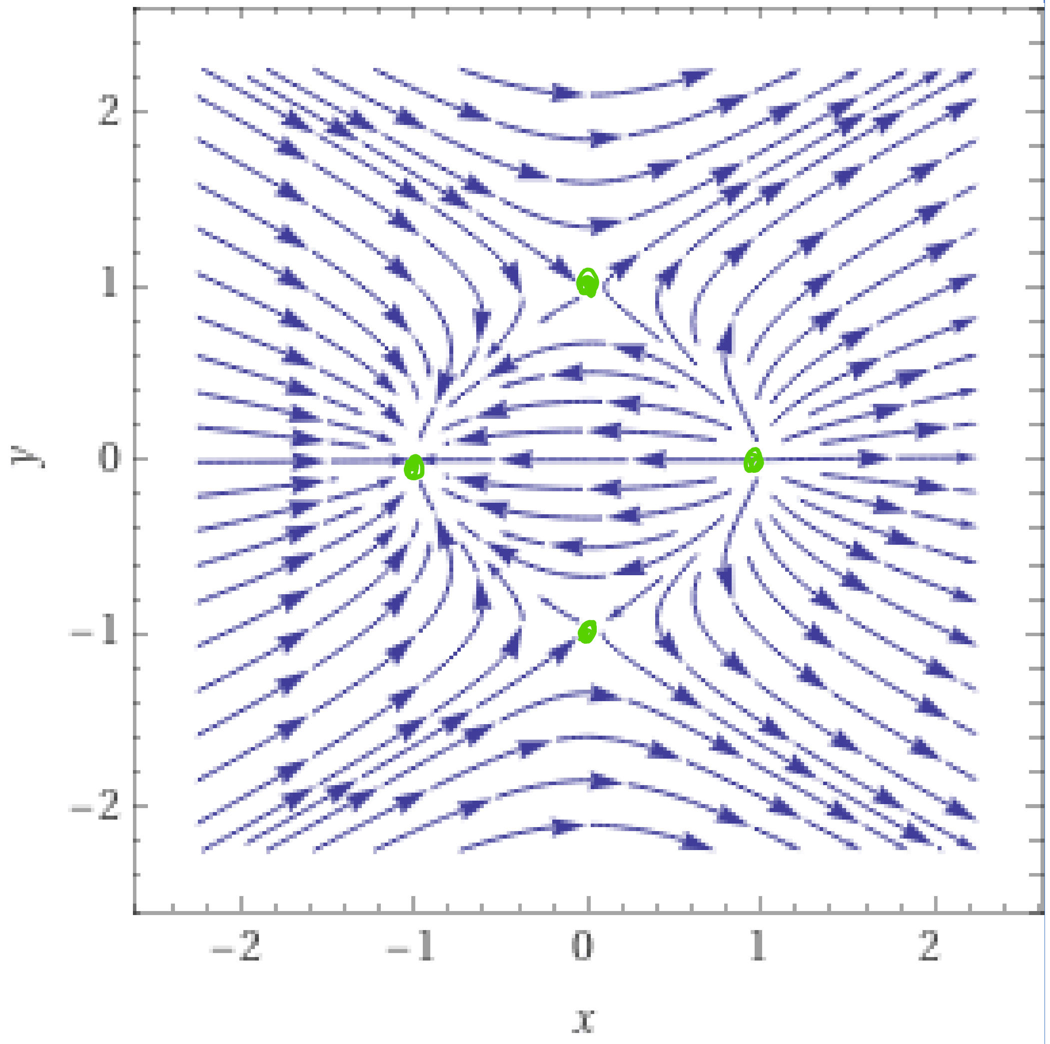
$\lambda = -2$, mult 2

proper node



Phase portraits





Try it out

- Find the equilibrium points for the nonlinear system:

$$\dot{x} = xy$$
$$\dot{y} = x + 2y - 8$$

A: (0, 4) and (8, 0)
B: (4,8) and (0,0)
C: (-4, 0) and (-8, 0)
D: (-4, -8) and (4, 8)
E: None of the above

$xy = 0$ $\quad$ $0 = x + 2y - 8$

Case 1: $x = 0$

$0 = 2y - 8 \Rightarrow y = 4$ $\quad$ $(0, 4)$

Case 2: $y = 0$

$0 = x - 8 \Rightarrow x = 8$ $\quad$ $(8, 0)$

Try it out

- Find the Jacobian matrix for the nonlinear system:

$$\dot{x} = xy$$
$$\dot{y} = x + 2y - 8$$

A: $\begin{bmatrix} x & y \\ x & 2y \end{bmatrix}$

B: $\begin{bmatrix} x & y \\ 1 & 2 \end{bmatrix}$

C: $\begin{bmatrix} x & y \\ 2y & x \end{bmatrix}$

D: $\begin{bmatrix} y & x \\ 1 & 2 \end{bmatrix}$

E: None of the above

$$J(x,y) = \begin{bmatrix} \frac{\partial}{\partial x}[xy] & \frac{\partial}{\partial y}[xy] \\ \frac{\partial}{\partial x}[x+2y-8] & \frac{\partial}{\partial y}[x+2y-8] \end{bmatrix}$$

$$= \begin{bmatrix} y & x \\ 1 & 2 \end{bmatrix}$$

Try it out

- Use the Jacobian to determine the behavior around the first equilibrium point:

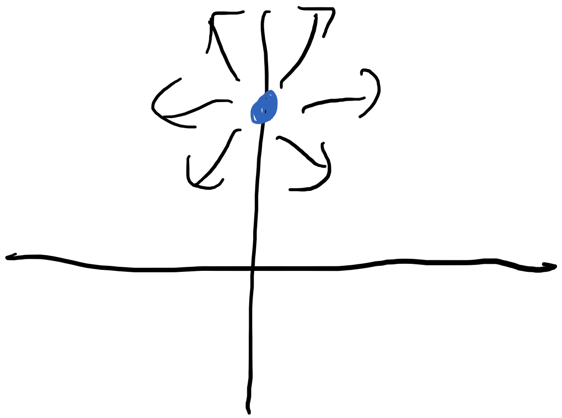
$$J(x,y) = \begin{bmatrix} y & x \\ 1 & 2 \end{bmatrix}$$

$$J(0,4) = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}$$

$$\lambda_1 = 4$$

$$\lambda_2 = 2$$

unstable



A: Asymptotically stable
B: Stable
C: Unstable
D: ???
E: None of the above

A: Node
B: Saddle point
C: Spiral
D: Center
E: None of the above

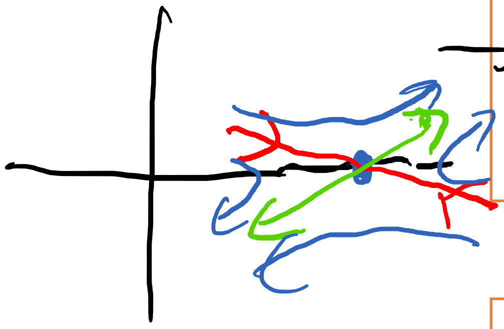
Try it out

- Use the Jacobian to determine the behavior around the ~~first~~ second equilibrium point:

(8, 0)

$$J(x,y) = \begin{bmatrix} y & x \\ 1 & 2 \end{bmatrix}$$

$$J(8,0) = \begin{bmatrix} 0 & 8 \\ 1 & 2 \end{bmatrix}$$



A: Asymptotically stable
B: Stable
C: Unstable
D: ???
E: None of the above

$$\begin{vmatrix} \lambda & -8 \\ -1 & \lambda-2 \end{vmatrix} = \lambda^2 - 2\lambda - 8 = 0$$

$$(\lambda+2)(\lambda-4) = 0$$

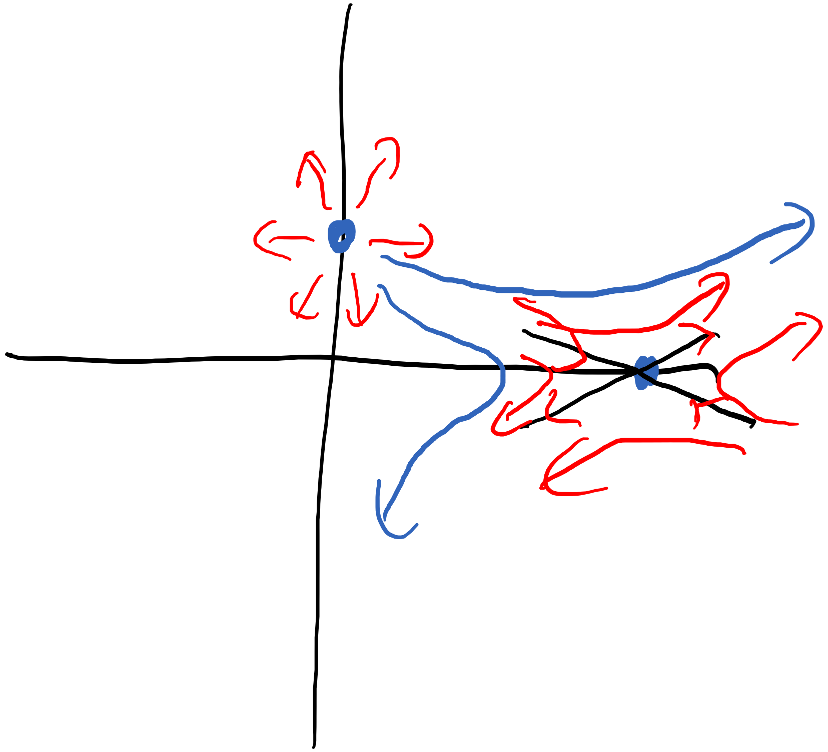
$$\lambda_1 = -2, \quad \lambda_2 = 4$$

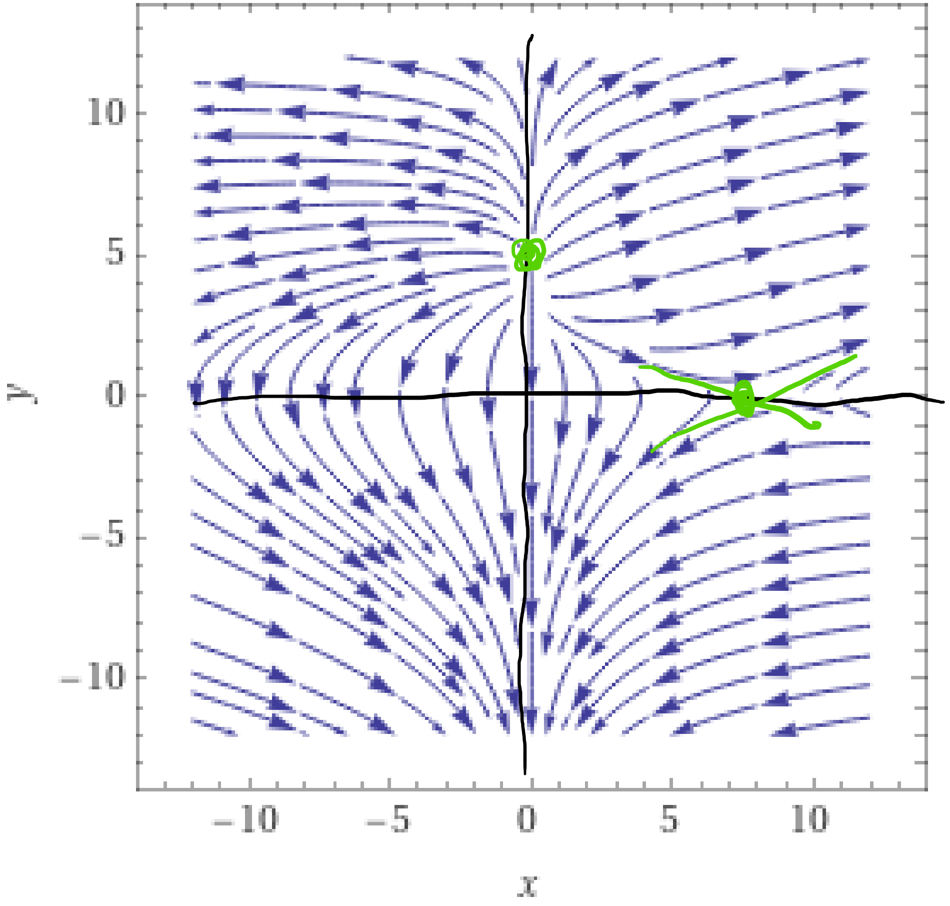
A: Node
B: Saddle point
C: Spiral
D: Center
E: None of the above

$\lambda_1 = -2 \qquad 8y = -2x \qquad v_1 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

$\lambda_2 = 4 \qquad 8y = 4x \qquad v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Putting it together





Predator-prey Lotka-Volterra model

- Consider an environment with wolves and deer.
- Deer population: $x(t)$
- Wolf population: $y(t)$
- The deer grow exponentially, but the population is kept in check by predation from the deer:

$$\dot{x}(t) = 2x - xy$$

interactions are bad for deer

- The wolves die out, unless they can find enough deer to eat.

$$\dot{y}(t) = -y + 0.4xy$$

interactions are good for wolves

Find equilibrium values and stability

- $\begin{cases} \dot{x} = 2x - xy \\ \dot{y} = -y + 0.4xy \end{cases}$

$$J(x,y) = \begin{bmatrix} 2-y & -x \\ 0.4y & -1+0.4x \end{bmatrix}$$

$0 = 2x - xy = x(2-y) \Rightarrow x=0$ or $y=2$

$0 = -y + 0.4xy = y(0.4x-1)$

Case 1: $x=0 \Rightarrow y=0$ $(0,0)$

Case 2: $y=2 \Rightarrow 0=0.4x-1 \Rightarrow x=2.5$ $(2.5, 2)$

$$J(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \left.\begin{matrix} \lambda_1 = 2 & v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \lambda_2 = -1 & v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{matrix}\right\} \text{saddle pt}$$

$$J(2.5,2) = \begin{bmatrix} 0 & -2.5 \\ 0.8 & 0 \end{bmatrix} \Rightarrow \lambda^2 + 2 = 0, \quad \lambda = \pm i\sqrt{2} \Rightarrow \text{center}$$

Phase diagram