Nonlinear Phase Portraits
Lecture 10d: 2021-07-28

MAT A35 – Summer 2021 – UTSC

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Summarizing everything

- $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
- The origin (0,0) is always an equilibrium point, and the eigenvalues of A determine the behavior at that equilibrium.
- Sign:
  - Positive real parts → trajectories go outward (unstable).
  - Negative real parts → trajectories go inward (asymptotically stable).
  - Opposite sign eigenvalues → saddle point (unstable).
- Real vs complex:
  - Real eigenvalues give either nodes (if both same sign) or saddle points.
  - Nonzero imaginary components mean that trajectories spiral.
  - Pure imaginary eigenvalues give rise to a “center”. (stable)

Try it out

- $\lambda_1 = 4, \lambda_2 = -2$
- $\lambda_1 = -3, \lambda_2 = -1$
- $\lambda_1 = 2, \lambda_2 = 3$
- $\lambda_1 = 3, \lambda_2 = 3$
- $\lambda_1 = 3 + 2i, \lambda_2 = 3 - 2i$
- $\lambda_{1,2} = -1 \pm 2i$
- $\lambda_{1,2} = \pm 4i$

A: Node (incl. (im)proper)
B: Saddle Point
C: Spiral
D: Center
E: None of the above

Special note: weird stuff can happen when $\lambda = 0$, which we won't deal with.

Example

- Classify the behavior around the origin of $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Example

- Classify the behavior around the origin of $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Try it out

- Classify the behavior around the origin of $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

A: Asymptotically Stable
B: Stable
C: Unstable
D: ???
E: None of the above

A: Node (incl. (im)proper)
B: Saddle Point
C: Spiral
D: Center
E: None of the above

Nonlinear autonomous systems and Jacobians

- $\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases}$
- Equilibrium points when $\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases}$
- We can approximate a function around a point using its derivative at that point.
- The *Jacobian* of the system is the analogue of the derivative:

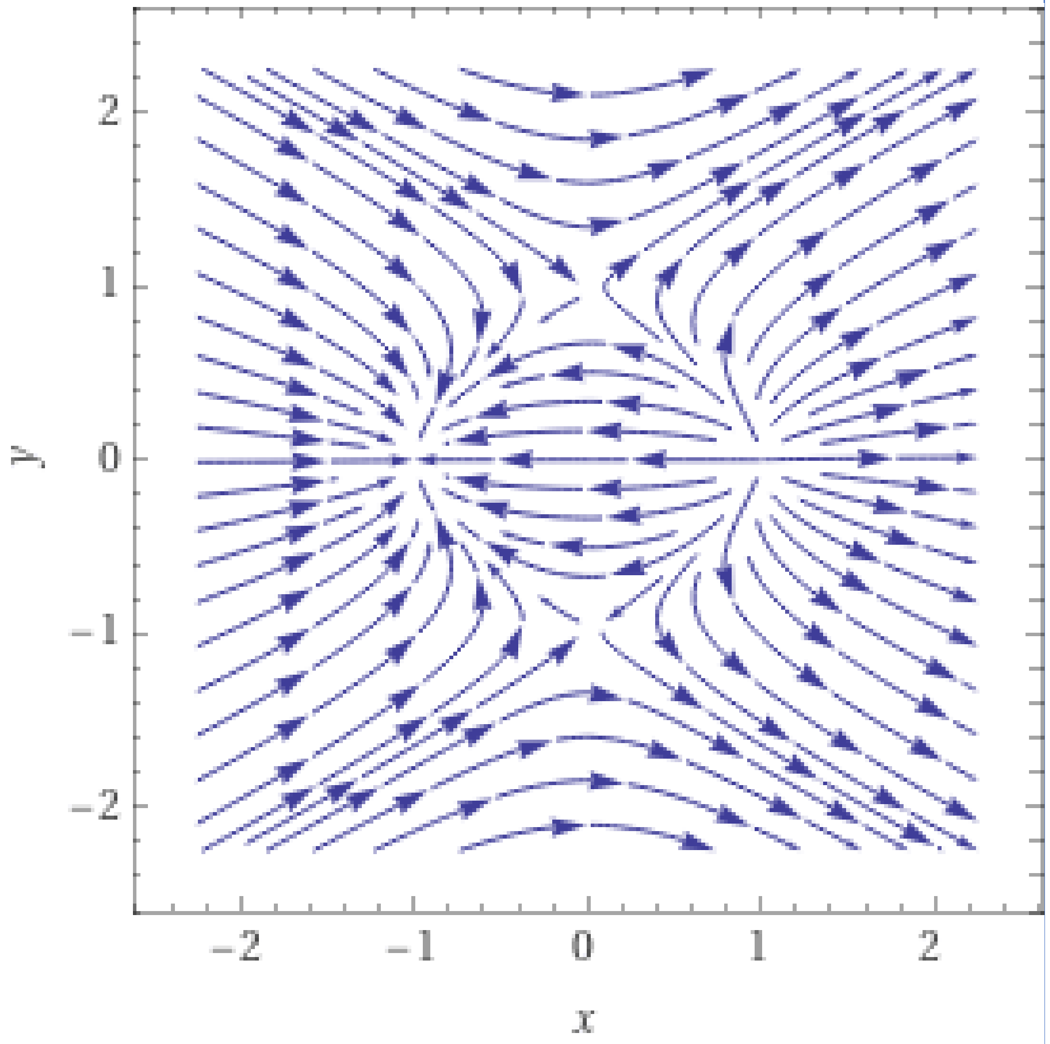
$$J(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

Nonlinear equilibria behavior

- Around each equilibrium, we can approximate its behavior by looking at the Jacobian matrix' eigenvalues.

Local phase portraits

Phase portraits



Try it out

- Find the equilibrium points for the nonlinear system:

$$\dot{x} = xy$$
$$\dot{y} = x + 2y - 8$$

A: (0, 4) and (8, 0)
B: (4,8) and (0,0)
C: (-4, 0) and (-8, 0)
D: (-4, -8) and (4, 8)
E: None of the above

Try it out

- Find the Jacobian matrix for the nonlinear system:

$$\dot{x} = xy$$
$$\dot{y} = x + 2y - 8$$

A: $\begin{bmatrix} x & y \\ x & 2y \end{bmatrix}$

B: $\begin{bmatrix} x & y \\ 1 & 2 \end{bmatrix}$

C: $\begin{bmatrix} x & y \\ 2y & x \end{bmatrix}$

D: $\begin{bmatrix} y & x \\ 1 & 2 \end{bmatrix}$

E: None of the above

Try it out

- Use the Jacobian to determine the behavior around the first equilibrium point:

A: Asymptotically stable
B: Stable
C: Unstable
D: ???
E: None of the above

A: Node
B: Saddle point
C: Spiral
D: Center
E: None of the above

Try it out

- Use the Jacobian to determine the behavior around the first equilibrium point:

A: Asymptotically stable
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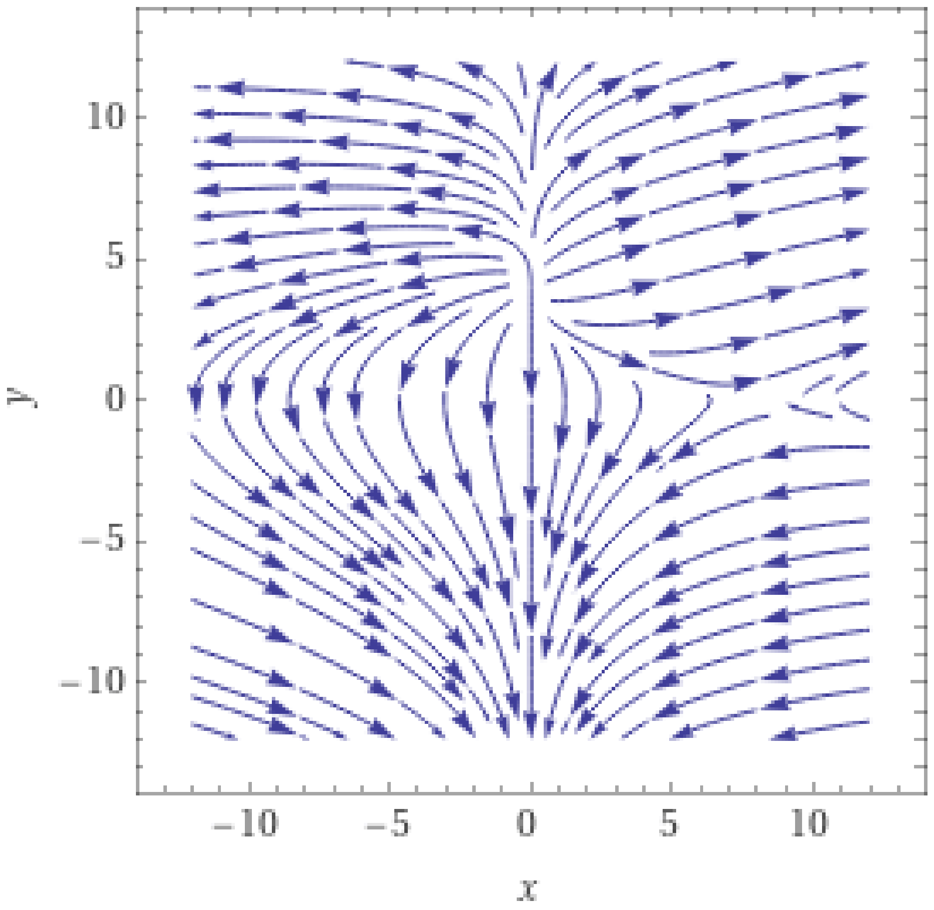
Putting it together



Predator-prey Lotka-Volterra model

- Consider an environment with wolves and deer.
- Deer population: $x(t)$
- Wolf population: $y(t)$
- The deer grow exponentially, but the population is kept in check by predation from the deer:
$$\dot{x}(t) = 2x - xy$$
- The wolves die out, unless they can find enough deer to eat.
$$\dot{y}(t) = -y + 0.4xy$$

Find equilibrium values and stability

- $\begin{cases} \dot{x} = 2x - xy \\ \dot{y} = -y + 0.4xy \end{cases}$

Phase diagram