Power series
Lecture 11b: 2021-08-04

MAT A35 – Summer 2021 – UTSC

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Simple mathematical operations

- Which mathematical operation is the hardest?
- Adding, subtracting, or multiplying two real numbers gives a real number.

$5+3=8$ $\quad$ $4 \cdot 0 = 0$

$3-5=-2$ $\quad$ $\pi \cdot e$ are real

- Dividing two real numbers may not.

$4/5 = 0.8$

$4/0 =$ not a number

A: Addition
B: Subtraction
C: Multiplication
D: Division
E: All are equally hard

Polynomials

- A real polynomial $p(x)$ in a variable x is an expression that combines together x with real numbers using just addition, subtraction, and multiplication, but no division.

poly: $p(x) = (x+1)(x-1) + 5x \cdot x - 3.4x + \pi$ ✓

not poly: $f(x) = \frac{1}{x-1}$ $f(x) = e^x$ $f(x) = (\ln x) x^2$

- Canonical form for nth-order polynomials:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_i \in \mathbb{R}$ and n is a positive integer.

Ex. $p(x) = x^2 - 1 + 5x^2 - 3.4x + \pi$

$p(x) = 6x^2 + (-3.4)x + (\pi - 1)$

Polynomials are nice

- Polynomials are built up from the “easy” operations of addition and multiplication (and implicitly subtraction).
- If you add together two polynomials, you get another polynomial.

$p(x) = x^2 + 2x + 1 \qquad q(x) = x^3 - 1 \qquad p(x) + q(x) = x^3 + x^2 + 2x$

- If you multiply together two polynomials, you get another polynomial.

$p(x)q(x) = x^5 + 2x^4 + x^3 - x^2 - 2x - 1$

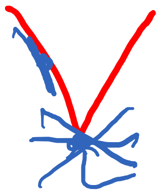
- Polynomials are infinitely “smooth” meaning you can keep on taking derivatives at any point.

$p'(x) = 2x + 2 \qquad p'''(x) = 0$

$p''(x) = 2 \qquad p''''(x) = 0$

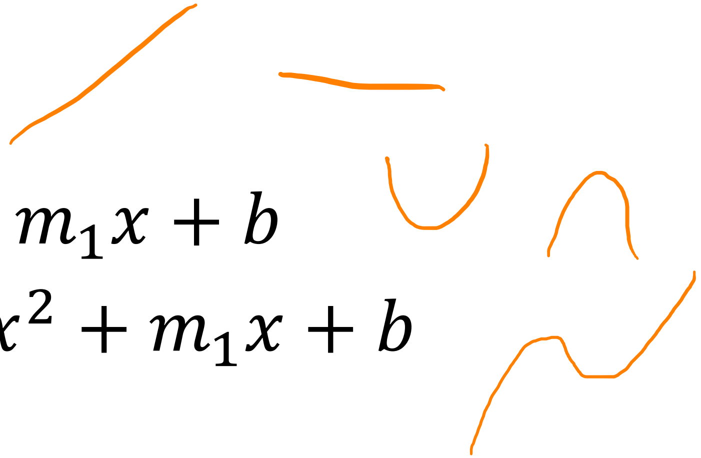
$\vdots$

nonexample: $f(x) = |x|$



$$f'(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \\ \text{undefined}, & x = 0 \end{cases}$$

Recall: different types of regression



- Linear regression: $f(x) = mx + b$
- Quadratic regression: $f(x) = m_2x^2 + m_1x + b$
- Cubic regression: $f(x) = m_3x^3 + m_2x^2 + m_1x + b$
- Polynomial regression of degree n:

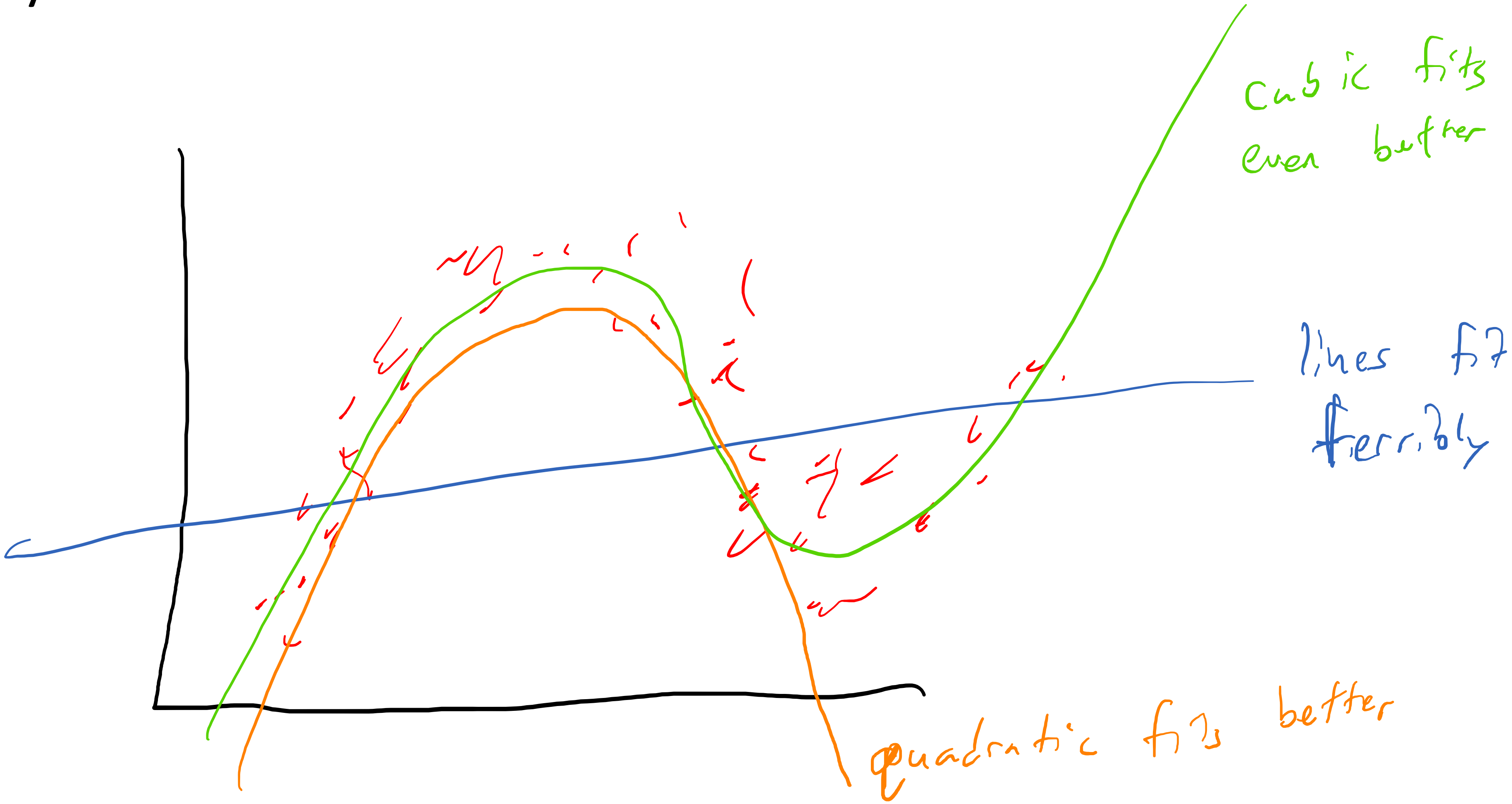
$$f(x) = b + \sum_{i=1}^{n} m_i x^i$$

can add more parameters for better fit

- Exponential regression: $f(x) = c_1 e^{c_2 x}$
- Power dependencies: $f(x) = c_1 x^{c_2}$

Recall: polynomial regression

- Given a collection of points, can approximate it with a polynomial function.



Be careful about too many parameters

- The more parameters you have (e.g. in a polynomial regression), the better your mean squared error will be.
- However, sometimes, you will overfit to the data.
- John von Neumann: “with four parameters, I can fit an elephant, and with five I can make him wiggle his trunk”.

Approximating non-polynomial functions

- Sometimes, another “nice” function looks almost like a polynomial, at least locally.

Ex.

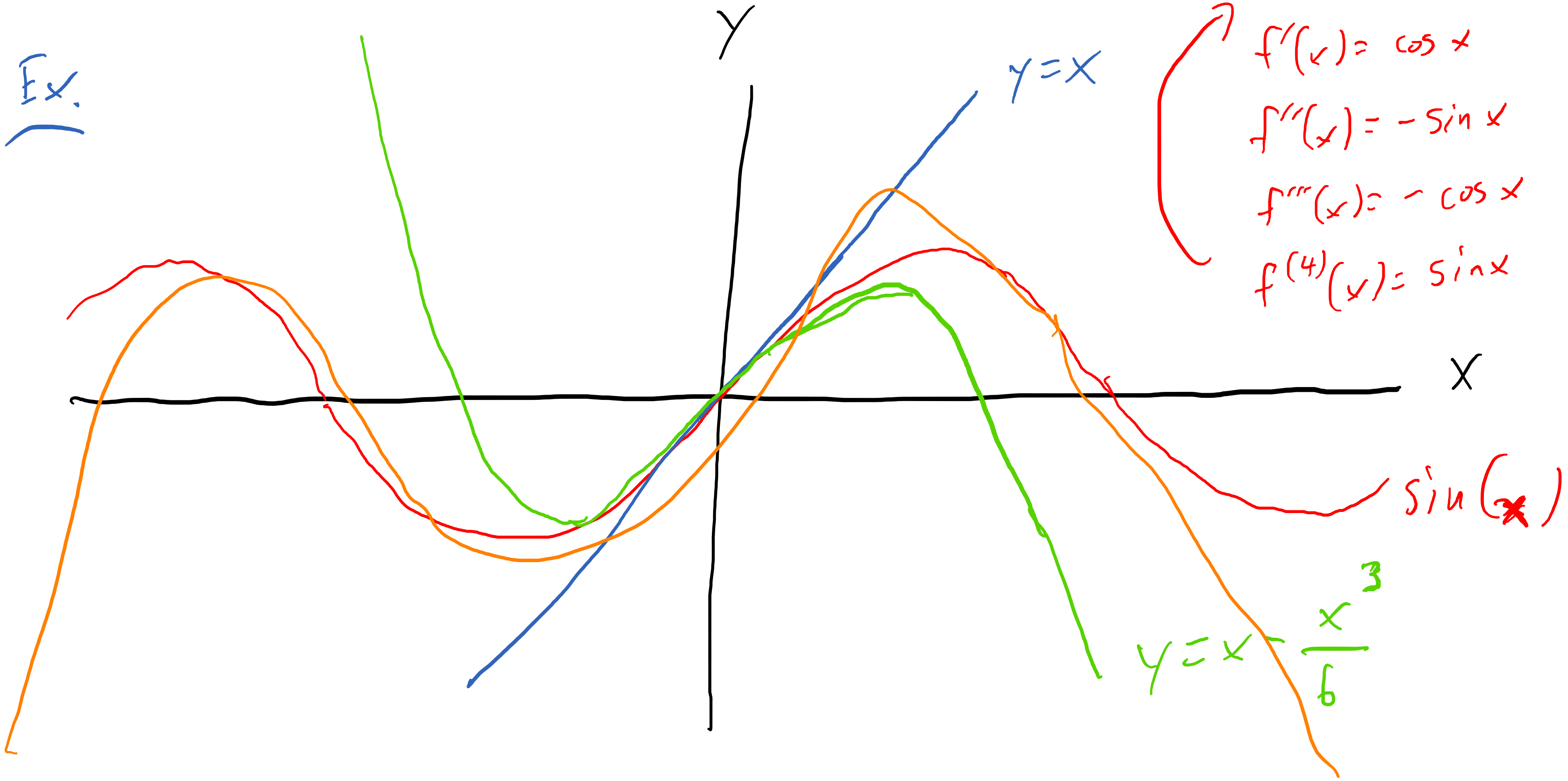
$f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{(4)}(x) = \sin x$



Formal power series

- A polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

can also be written as

$$p(x) = \sum_{i=0}^{n} a_i x^i$$

Ex. $p(x) = 1 + x + x^2 + x^3$

- A formal power series is the infinite sum where $n \to \infty$

$$p(x) = \sum_{i=0}^{\infty} a_i x^i$$

Ex. $p(x) = 1 + x + x^2 + x^3 + \cdots = \sum_{i=0}^{\infty} x^i$

Convergence

$$\sum_{i=0}^{\infty} a_i x^i = \lim_{n\to\infty} \sum_{i=0}^{n} a_i x^i$$

- A formal power series $p(x) = \sum_{i=0}^{\infty} a_i x^i$ converges at a value x if the infinite sequence $x_0, x_1, x_2, \ldots$, where $x_n = \sum_{i=0}^{n} a_i x^i$ converges to a limit as $n \to \infty$. It is divergent otherwise.

Ex.

$$p(x) = \sum_{i=0}^{\infty} x^i = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$p\left(\tfrac{1}{2}\right) = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots = 2$$

$$p(1) = 1 + 1 + 1 + 1 + \cdots$$ divergent

If $|x| < 1$, $p(x) = \dfrac{1}{1-x}$ (Geometric Series)

If $|x| \geq 1$, $p(x)$ is divergent

$p(-1) =$ [illegible] $1 - 1 + 1 - 1 + 1 - 1 \cdots$ divergent

A: $p(0.5) = 1$
B: $p(0.5) = 2$
C: $p(0.5) = 3$
D: $p(0.5)$ is divergent
E: None of the above

A: $p(1) = 1$
B: $p(1) = 2$
C: $p(1) = 3$
D: $p(1)$ is divergent
E: None of the above

Power series for a function

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = \frac{1}{x}$

- A formal power series $p(x) = \sum_{i=0}^{\infty} a_i x^i$ can be thought of as a function whose domain is the interval of convergence.

Ex. $p(x) = 1 + x + x^2 + x^3 + x^4 + \cdots$ is a function on the domain $(-1, 1)$

- Sometimes, we can express another function as a power series, at least on some interval of convergence.

Ex. $x^2 + 2x + 1$ is a polynomial and already a power series

$x^2 + 2x + 1 = \sum_{i=0}^{\infty} a_i x^i$, where $a_0 = 1$, $a_1 = 2$, $a_2 = 1$, $a_3 = 0, \ldots$

Ex. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$ on the interval $(-1, 1)$

If $x \in (-1, 1)$, $\frac{1}{1-x} = 1 + x + x^2 + \cdots$

Manipulating power series

- How do we come up with a power series for a function?

Ex. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ for $|x| < 1$

- Sometimes, we can manipulate it algebraically.

Multiply by $(1-x)$

Check: $1 = (1-x) + x(1-x) + x^2(1-x) + x^3(1-x) + \cdots$

$1 = (1-x) + (x - x^2) + (x^2 - x^3) + (x^3 - x^4) + \cdots$

$1 = 1 + \underline{(-x + x)} + \underline{(-x^2 + x^2)} + \underline{(-x^3 + x^3)} + \cdots \to 0$

✓

Ex. $p(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$

$p(x^3) = \frac{1}{1-x^3} = 1 + x^3 + x^6 + x^9 + x^{12} + \cdots$

More examples

In the interval $(-1, 1)$

- Adding together power series

$$p(x) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

$$h(x) = \frac{1}{1-x^3} = 1 + x^3 + x^6 + x^9 + \cdots$$

$$\frac{1}{1-x} + \frac{1}{1-x^3}$$

$$= 2 + x + x^2 + 2x^3 + x^4 + x^5 + 2x^6 + x^7 + \cdots$$

- Multiplication of a power series by a polynomial

$$g(x) = \frac{x}{1-x} = x \cdot \left(\frac{1}{1-x}\right) = x + x^2 + x^3 + x^4 + \cdots$$

Try it out

- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{i=0}^{\infty} x^i$
- What is $\frac{1}{1-2x}$? $= 1 + 2x + (2x)^2 + (2x)^3 + (2x^4) + \cdots$

$= 1 + 2x + 4x^2 + 8x^3 + 16x^4 + \cdots$

$= \sum_{i=0}^{\infty} (2x)^i$

$= \sum_{i=0}^{\infty} 2^i x^i$

A: $\sum_{i=0}^{\infty} 2x^i$
B: $\sum_{i=0}^{\infty} 2^i x^i$
C: $\sum_{i=0}^{\infty} \frac{2^i x^i}{i!}$
D: $\sum_{i=0}^{\infty} \frac{x^{2i}}{i!}$
E: None of the above

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$ ← will derive soon
- What is e^{x^2}?

$= 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots = \sum_{i=0}^{\infty} \frac{x^{2i}}{i!}$

Taylor series intuition

- If the space shuttle is moving at 10 m/s away from Earth, how far away from Earth is it after 1 minute?
- What if its speed is not constant?
- If the space shuttle is moving at 10 m/s, and it is constantly accelerating at 1 m/s^2, how far away is it after 1 minute?

A: 10 meters
B: 600 meters
C: 1000 meters
D: 2400 meters
E: None of the above

$\int \ddot{x}\,dt = \int 1\,dt \rightarrow \dot{x}(t) = t + C$

$\dot{x}(0) = 10$ $\quad \dot{x}(0) = 10 \Rightarrow C = 0$

plug in 0

$x(0) = 0$ $\quad \int \dot{x}(t)\,dt = \int t + 10\,dt$

$0 = x(0) = C_2$ $\quad x(t) = \frac{1}{2}t^2 + 10t + C_2$ $\Rightarrow x(t) = \frac{1}{2}t^2 + 10t$

$x(60) = 1800 + 600 = 2400$

- What if its acceleration is not constant?

Taylor series intuition (part 2)

- If we know all the derivatives of a polynomial at a point (e.g. at x=0), then we can reconstruct the polynomial.

Ex.

$$p(x) = 4 + 3x + 2x^2 + x^3$$

$$p'(x) = 3 + 4x + 3x^2$$

$$p''(x) = 4 + 6x$$

$$p'''(x) = 6$$

$$p''''(x) = 0$$

$$\vdots$$

$$p^{(n)}(x) = 0, \quad n \geq 4$$

$$p(0) = 4$$

$$p'(0) = 3$$

$$p''(0) = 4$$

$$p'''(0) = 6$$

$$\left.\begin{array}{l} p^{(4)}(0) = 0 \\ \vdots \\ p^{(n)}(0) = 0 \end{array}\right\} \text{order of poly is } 3$$

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$p'(x) = a_1 + 2a_2 x + 3a_3 x^2$$

$$p''(x) = 2a_2 + 6a_3 x$$

$$p'''(x) = 6a_3$$

$$p(0) = a_0$$

$$p'(0) = a_1$$

$$p''(0) = 2a_2$$

$$p'''(0) = 6a_3$$

$$\Rightarrow a_0 = 4$$

$$a_1 = 3$$

$$a_2 = 2$$

$$a_3 = 1$$

Taylor and Maclaurin series definitions

- The Maclaurin series of a function $f(x)$ is given by
$$\sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots$$

- The Taylor series of a function $f(x)$ at a real number a is the power series
$$\sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!} (x-a)^i$$
- The Maclaurin series is just the Taylor series at $a = 0$, and is the power series with matching derivatives at 0 with the original function $f(x)$.

Asides

- The Taylor series for any polynomial is the polynomial itself.
- A Taylor series may not necessarily converge at a point even if the function is well defined.
- A function may differ from the sum of its Taylor series, even if the Taylor series is convergent.
- However, for many common functions, the function and the sum of its Taylor series are equal in some radius of convergence.

e^x, $\sin x$, $\cos x$

Examples

$$= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

- $f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$

Note: $\frac{d}{dx} e^x = e^x$, so $f^{(i)}(0) = 1$

Thus, the Maclaurin series is

$$\sum_{i=0}^{\infty} \frac{f^{i}(0)}{i!} x^i = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

$$\frac{d}{dx}\Big[\underbrace{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}_{e^x}\Big] = 0 + \underbrace{1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}_{e^x}$$

Example

$$= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

- What is the Maclaurin series for $f(x) = \cos x$?

$$\sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i = 1 + \frac{0}{1!}x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \dots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \dots$$

$$f(x) = \cos x \qquad f(0) = 1$$

$$f'(x) = -\sin x \qquad f'(0) = 0$$

$$f''(x) = -\cos x \qquad f''(0) = -1$$

$$f'''(x) = \sin x \qquad f'''(0) = 0$$

$$\vdots \qquad f^{(4)}(0) = 1$$

$$0$$

$$-1$$

$$\vdots$$

$$= \sum_{i=0}^{\infty} (-1)^i \cdot \frac{x^{2i}}{(2i)!}$$

Try it out

- What is the Maclaurin series for $f(x) = \sin x$?

$$\sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots$$

$$= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots$$

$f(x) = \sin x$ $\qquad f(0) = 0$

$f'(x) = \cos x$ $\qquad f'(0) = 1$

$f''(x) = -\sin x$ $\qquad f''(0) = 0$

$f'''(x) = -\cos x$ $\qquad f'''(0) = -1$

$f^{(4)}(x) = \sin x$

$\vdots$ $\qquad \vdots$

A: $1 + x + x^2 + x^3 + x^4 + \cdots$

B: $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

C: $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$

D: $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$

E: None of the above

Proof of Euler's Equation

- $e^{ix} = \cos x + i \sin x$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots$$

$$= 1 + ix + -\frac{1}{2!}x^2 - i\frac{x^3}{3!} + \frac{x^4}{4!}$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= \cos x + i \sin x$$