Power series
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Simple mathematical operations

- Which mathematical operation is the hardest?
- Adding, subtracting, or multiplying two real numbers gives a real number.
- Dividing two real numbers may not.

A: Addition
B: Subtraction
C: Multiplication
D: Division
E: All are equally hard

Polynomials

- A real polynomial $p(x)$ in a variable x is an expression that combines together x with real numbers using just addition, subtraction, and multiplication, but no division.

- Canonical form for nth-order polynomials:
$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
where $a_i \in \mathbb{R}$ and n is a positive integer.

Polynomials are nice

- Polynomials are built up from the “easy” operations of addition and multiplication (and implicitly subtraction).
- If you add together two polynomials, you get another polynomial.
- If you multiply together two polynomials, you get another polynomial.
- Polynomials are infinitely “smooth” meaning you can keep on taking derivatives at any point.

Recall: different types of regression

- Linear regression: $f(x) = mx + b$
- Quadratic regression: $f(x) = m_2x^2 + m_1x + b$
- Cubic regression: $f(x) = m_3x^3 + m_2x^2 + m_1x + b$
- Polynomial regression of degree n:

$$f(x) = b + \sum_{i=1}^{n} m_i x^i$$

- Exponential regression: $f(x) = c_1 e^{c_2 x}$
- Power dependencies: $f(x) = c_1 x^{c_2}$

Recall: polynomial regression

- Given a collection of points, can approximate it with a polynomial function.

Be careful about too many parameters

- The more parameters you have (e.g. in a polynomial regression), the better your mean squared error will be.
- However, sometimes, you will overfit to the data.
- John von Neumann: “with four parameters, I can fit an elephant, and with five I can make him wiggle his trunk”.

Approximating non-polynomial functions

- Sometimes, another “nice” function looks almost like a polynomial, at least locally.

Formal power series

- A polynomial
$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
can also be written as
$$p(x) = \sum_{i=0}^{n} a_i x^i$$
- A formal power series is the infinite sum where $n \to \infty$
$$p(x) = \sum_{i=0}^{\infty} a_i x^i$$

Convergence

- A formal power series $p(x) = \sum_{i=0}^{\infty} a_i x^i$ converges at a value x if the infinite sequence $x_0, x_1, x_2, \ldots,$ where $x_n = \sum_{i=0}^{n} a_i x^i$ converges to a limit as $n \to \infty$. It is divergent otherwise.

A: $p(0.5) = 1$
B: $p(0.5) = 2$
C: $p(0.5) = 3$
D: $p(0.5)$ is divergent
E: None of the above

A: $p(1) = 1$
B: $p(1) = 2$
C: $p(1) = 3$
D: $p(1)$ is divergent
E: None of the above

Power series for a function

- A formal power series $p(x) = \sum_{i=0}^{\infty} a_i x^i$ can be thought of as a function whose domain is the interval of convergence.

- Sometimes, we can express another function as a power series, at least on some interval of convergence.

Manipulating power series

- How do we come up with a power series for a function?

- Sometimes, we can manipulate it algebraically.

Try it out

- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{i=0}^{\infty} x^i$
- What is $\frac{1}{1-2x}$?

A: $\sum_{i=0}^{\infty} 2x^i$
B: $\sum_{i=0}^{\infty} 2^i x^i$
C: $\sum_{i=0}^{\infty} \frac{2^i x^i}{i!}$
D: $\sum_{i=0}^{\infty} \frac{x^{2i}}{i!}$
E: None of the above

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$
- What is e^{x^2}?

Taylor series intuition

- If the space shuttle is moving at 10 m/s away from Earth, how far away from Earth is it after 1 minute?
- What if its speed is not constant?
- If the space shuttle is moving at 10 m/s, and it is constantly accelerating at 1 m/s^2, how far away is it after 1 minute?

A: 10 meters
B: 600 meters
C: 1000 meters
D: 2400 meters
E: None of the above

- What if its acceleration is not constant?

Taylor series intuition (part 2)

- If we know all the derivatives of a polynomial at a point (e.g. at x=0), then we can reconstruct the polynomial.

Taylor and Maclaurin series definitions

- The Maclaurin series of a function $f(x)$ is given by
$$\sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots$$

- The Taylor series of a function $f(x)$ at a real number a is the power series
$$\sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!} (x-a)^i$$
- The Maclaurin series is just the Taylor series at $a = 0$, and is the power series with matching derivatives at 0 with the original function $f(x)$.

Asides

- The Taylor series for any polynomial is the polynomial itself.
- A Taylor series may not necessarily converge at a point even if the function is well defined.
- A function may differ from the sum of its Taylor series, even if the Taylor series is convergent.
- However, for many common functions, the function and the sum of its Taylor series are equal in some radius of convergence.

Examples

- $f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$

Example

- What is the Maclaurin series for $f(x) = \cos x$?

Try it out

- What is the Maclaurin series for $f(x) = \sin x$?

A: $1 + x + x^2 + x^3 + x^4 + \cdots$

B: $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

C: $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$

D: $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$

E: None of the above

Proof of Euler's Equation

- $e^{ix} = \cos x + i \sin x$