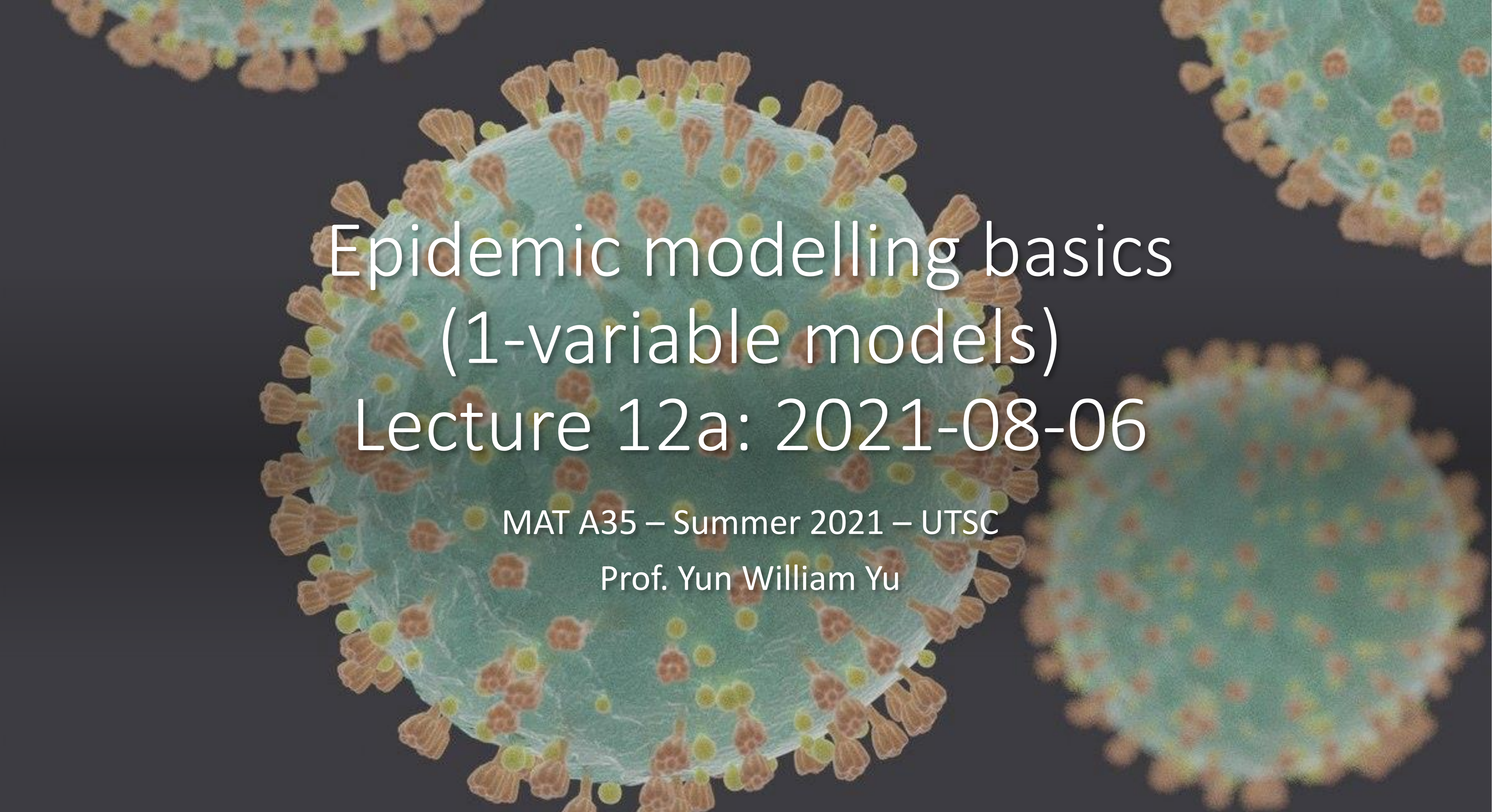


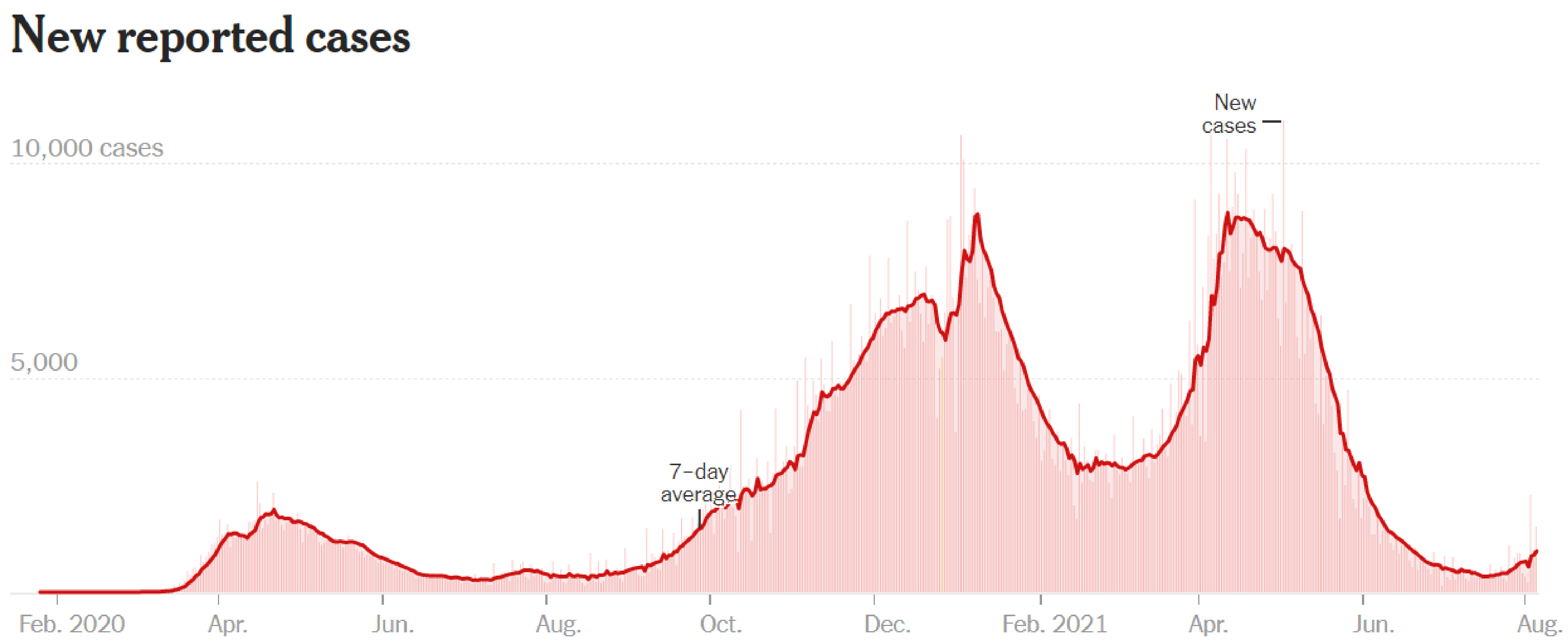
Epidemic modelling basics (1-variable models) Lecture 12a: 2021-08-06

MAT A35 – Summer 2021 – UTSC

Prof. Yun William Yu

Image credit: https://commons.wikimedia.org/wiki/File:Coronavirus_SARS-CoV-2.jpg

Epidemic curves – Covid19 in Canada



https://www.nytimes.com/interactive/2021/world/canada-covid-cases.html

Infection rates

- Assumption 1: each infected individual infects other individuals at a constant positive rate β.

Model 1: Let $I(t) = \#$ infected at time t.

$$\frac{dI}{dt} = \beta I \implies \dot{I} = \beta I$$

$$\dot{I} - \beta I = 0$$

Char. eq. $\lambda - \beta = 0$

$$\implies \lambda = \beta$$

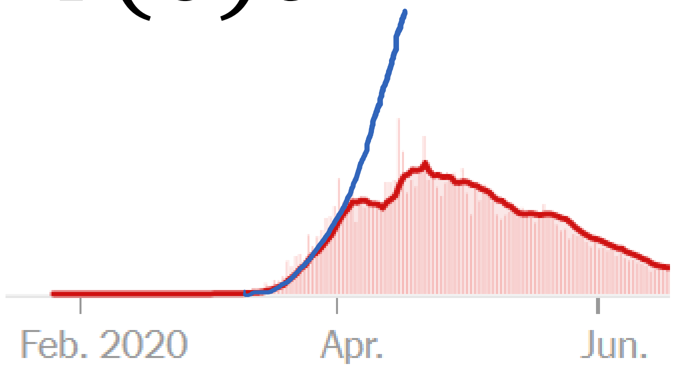
$$I(t) = Ce^{\beta t}$$

$$\implies I(0) = Ce^{\beta \cdot 0}$$

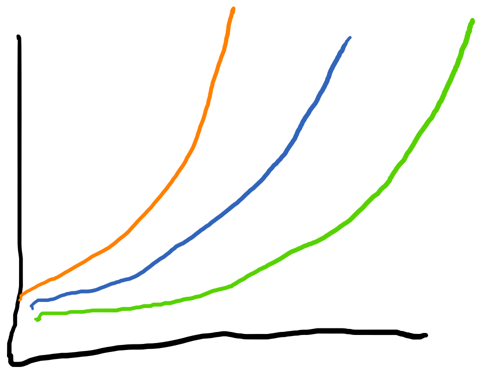
$$\implies C = I(0)$$

$$I(t) = I(0)\, e^{\beta t}$$

Exponential model: $I(t) = I(0)e^{\beta t}$



- Let's focus on the initial months of the pandemic.
- Can use regression to find the good values for β, or even just trial and error.



- Model doesn't take into account finite population size.

What's wrong with the model?
A: Model is too simple
B: Cannot determine good β
C: Doesn't reflect the data
D: All of the above
E: None of the above

Compartmental models

- Assumption 2: there is a total fixed population size $N = I(t) + S(t)$, where S is the number of Susceptible individuals.

$$\left.\begin{array}{l} \dot{I} = \beta I \\ N = I + S \end{array}\right\} \quad \begin{array}{l} \dot{I} = \beta I \\ S = N - I \\ \dot{S} = -\dot{I} = -\beta I \end{array}$$

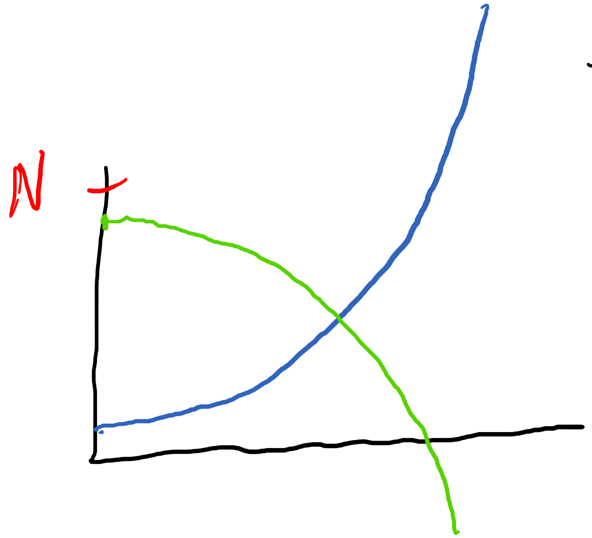
$$\boxed{S} \xrightarrow{\beta I} \boxed{I}$$

A: Yes
B: No
C: Maybe
D: ???

- Does this fix the problems from the previous slide?

$$\dot{I} = \beta I \implies I(t) = \underline{I(0) e^{\beta t}}$$

$$S(t) = N - I(0) e^{\beta t}$$



Model still goes off to infinite infected, and even worse, negative Susceptible

SI Model of Epidemics

- Modified assumption 1: The infection rate is proportional to the average number of times an infected individual encounters a susceptible individual in the population, assuming random encounters.

$$\beta \cdot I \cdot \frac{S}{N} = \text{infection rate}$$

$$\boxed{S} \xrightarrow{\frac{\beta S I}{N}} \boxed{I}$$

$$N = S(t) + I(t)$$

$$\dot{S} = -\frac{\beta S I}{N}$$

$$\dot{I} = \frac{\beta S I}{N}$$

Solving SI model qualitatively

- $N = S(t) + I(t)$
- $\dot{S} = -\dfrac{\beta S I}{N}$
- $\dot{I} = \dfrac{\beta S I}{N}$

$$S = N - I$$

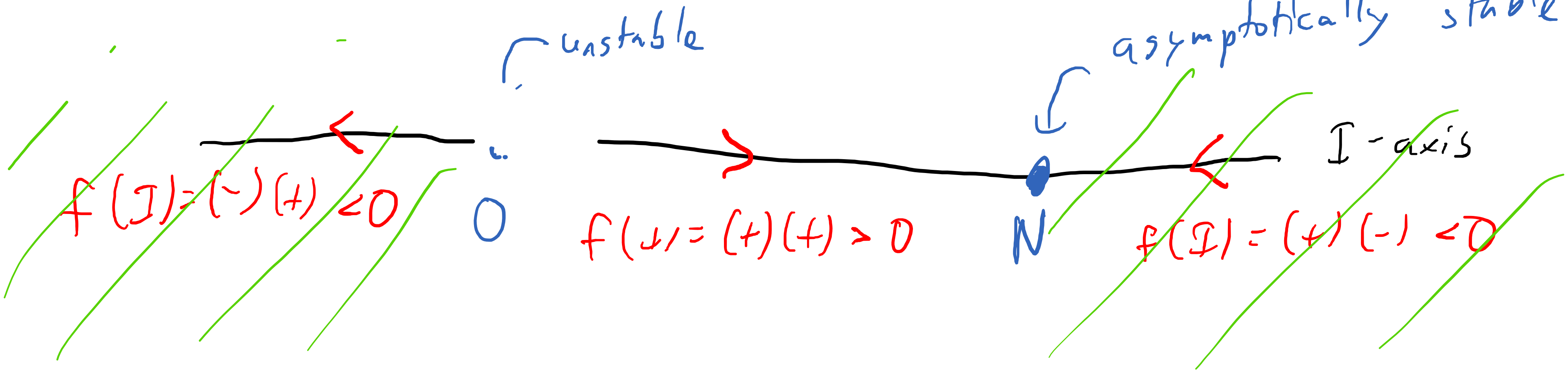
$$\dot{I} = \frac{\beta I}{N}(N - I) = f(I)$$

single-var. ODE

Equilibria:

$$0 = \dot{I} = \frac{\beta I}{N}(N - I)$$

$$\Rightarrow I = 0 \quad \text{or} \quad I = N$$



Solving the SI model exactly

- $\dot{I} = \frac{\beta I}{N}(N - I)$
- What methods should we use?

A: Separation of variables
B: Integrating factor
C: u-substitution
D: All of the above
E: None of the above

Bernoulli ODE, characteristic equation

~~Integrating factor~~ and u-substitution

- $\dot{I} = \frac{\beta I}{N}(N - I)$

$$\dot{I} = \beta I - \frac{\beta I^2}{N}$$

$$\left.\dot{I} - \beta I = -\frac{\beta I^2}{N}\right\}$$ Bernoulli ODE

Multiply by $-I^{-2}$

$$\frac{dI}{dt}\cdot\frac{-1}{I^2} + \frac{\beta}{I} = \frac{\beta}{N}$$

Let $u = \frac{1}{I}$, $du = \frac{-1}{I^2}dI$

$$I = \frac{1}{u}$$

$$\left.\frac{du}{dt} + \beta u = \frac{\beta}{N}\right\}$$ 1st-order linear

Could use I.F.

$$\lambda + \beta = 0 \Rightarrow \lambda = -\beta$$

$$u_h = Ce^{-\beta t}$$

Ansatz: $u_p = A$, $\dot{u}_p = 0$

$$\Rightarrow \beta A = \frac{\beta}{N}, \quad A = \frac{1}{N} \Rightarrow u_p = \frac{1}{N}$$

$$u_g = Ce^{-\beta t} + \frac{1}{N}$$

$$I = \frac{1}{Ce^{-\beta t} + \frac{1}{N}} = \boxed{\frac{N}{1 + Ce^{-\beta t}}}$$

Separation of variables

$$\log A - \log B = \log \frac{A}{B}$$

- $\dot{I} = \frac{\beta I}{N}(N - I) = \beta I \left(1 - \frac{I}{N}\right)$

$$\frac{dI}{dt} = \beta I \left(1 - \frac{I}{N}\right)$$

separate var.

$$\frac{dI}{I\left(1 - \frac{I}{N}\right)} = \beta\, dt$$

$$\frac{N}{I(N-I)}\, dI = \beta\, dt$$

partial fractions

$$\left[\frac{A}{I} + \frac{B}{N-I}\right] dI = \beta\, dt$$

$$A(N-I) + BI = N$$

$$AN + I(B-A) = N$$

$$\Rightarrow A = 1, \quad B = 1$$

$$\int \left[\frac{1}{I} + \frac{1}{N-I}\right] dI = \int \beta\, dt$$

$$-\ln|I| + \ln|N-I| = -\beta t + C$$

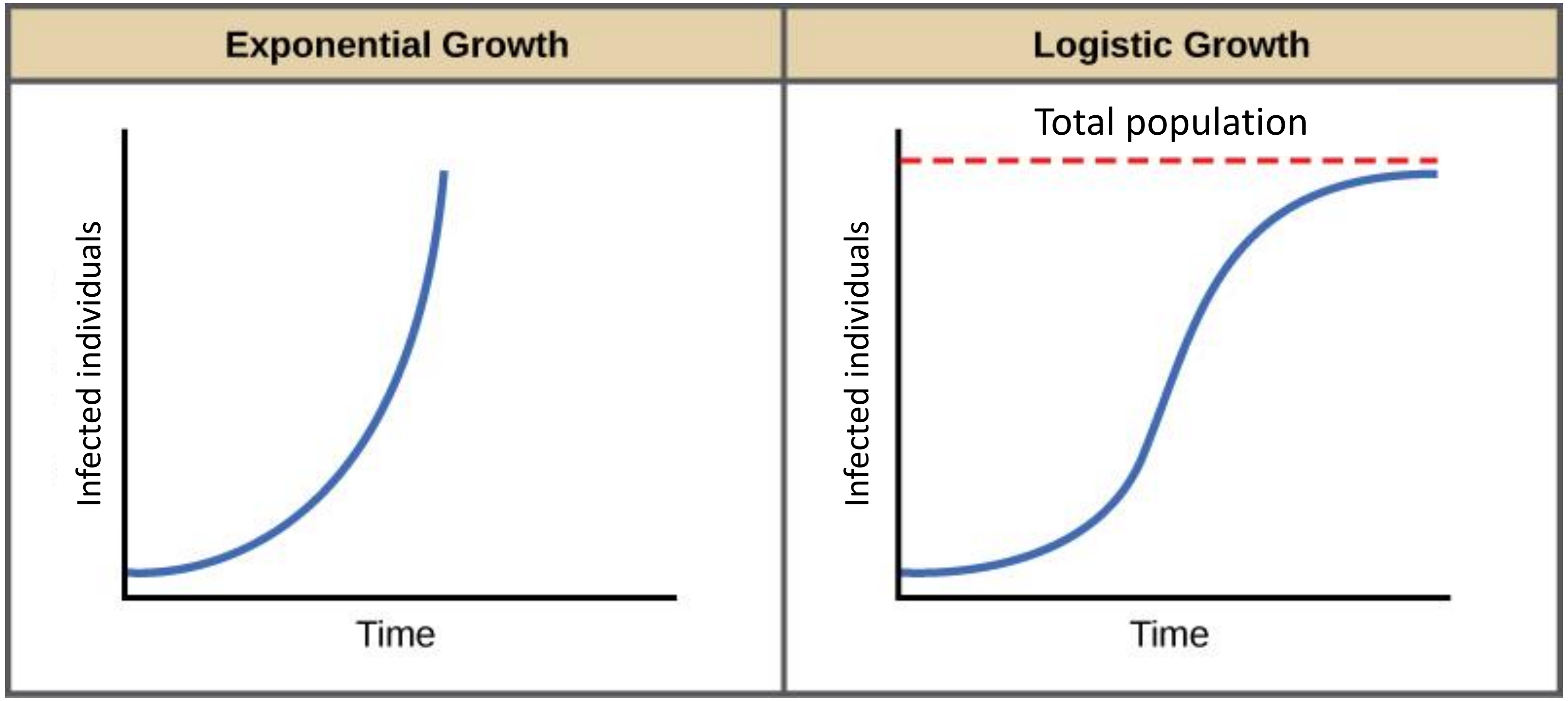
$$\ln\left|\frac{N-I}{I}\right| = -\beta t + C$$

$$\frac{N}{I} - 1 = \frac{N-I}{I} = Ce^{-\beta t}$$

$$\frac{N}{I} = 1 + Ce^{-\beta t} \Rightarrow \boxed{I = \frac{N}{1 + Ce^{-\beta t}}}$$

Logistic growth equation

- SI model has logistic growth, which starts out like exponential growth, but levels out as everyone is infected.

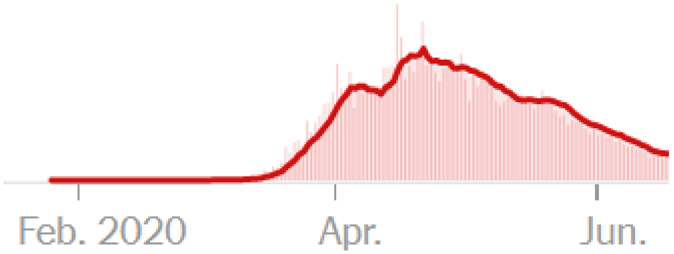


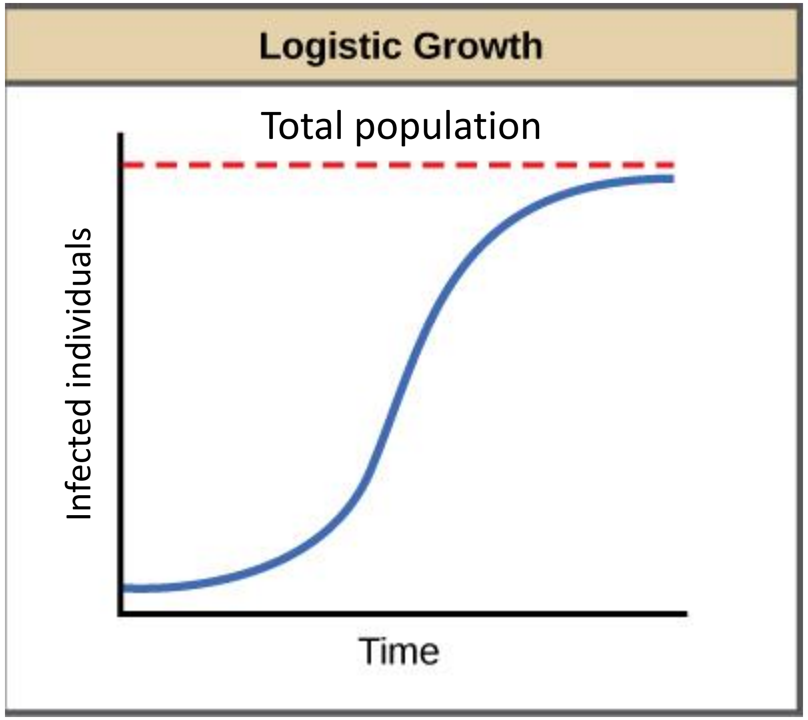
https://commons.wikimedia.org/wiki/File:Figure_45_03_01.jpg

Further improvements

- We now no longer go off to infinity, which is good.
- But, we still are missing the downward part of the epidemic curve.
- How do we get the number of infected to go back down in our model?

A: Add a recovery term
B: Add a mortality rate
C: Add an immunization rate
relevant
D: All of the above
E: None of the above





SIS Model

- Assumption 3: Individuals recover at rate γ, and become susceptible to re-infection.

$$S \underset{\gamma I}{\overset{\frac{\beta S I}{N}}{\rightleftarrows}} I$$

$$S(t) + I(t)$$

$$^{3}\ SI + \gamma I$$

$$\dot{I} = \frac{\beta}{N} S I - \gamma I$$

$$S = N - I$$

$$\dot{I} = \frac{\beta}{N}(N - I) I - \gamma I$$

$$\dot{I} = \left(\beta - \frac{\beta I}{N} - \gamma\right) I$$

Multiple cases

- $\frac{dI}{dt} = \left(\beta - \frac{\beta I}{N} - \gamma\right) I$
- Note, we have two parameters, β and γ, so there are three cases to consider.
- What are your guesses for behavior in each of the following?
- Case 1: $\beta = \gamma$
- Case 2: $\beta < \gamma$
- Case 3: $\beta > \gamma$

A: Disease dies out
B: Number of infected goes to nonzero constant
C: Number of infected oscillates up and down
D: All of the above
E: None of the above

Case 1: $\beta = \gamma$

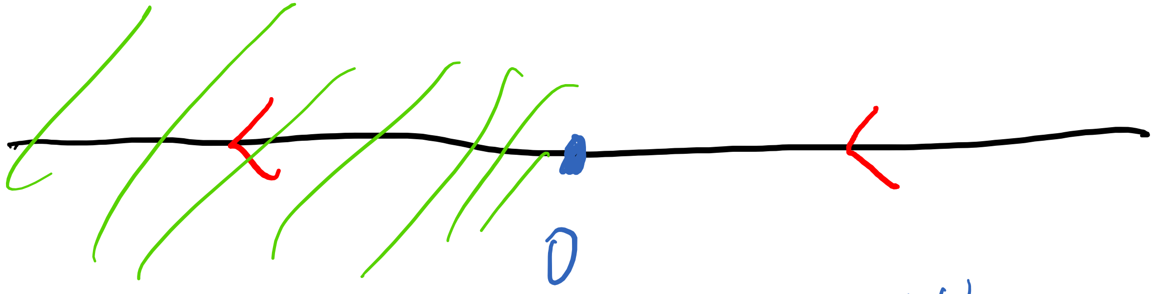
Infection rate = recovery rate

- $\frac{dI}{dt} = \left(\beta - \frac{\beta I}{N} - \gamma\right) I$

$\Rightarrow \dot{I} = -\frac{\beta I^2}{N}$

Eq. $\dot{I} = 0 \Rightarrow I = 0$

Note $-\frac{\beta I^2}{N} < 0$ for all $I \neq 0$



semi-stable

$\Rightarrow$ disease dies to 0 asymptotically

Case 2: $\beta < \gamma$

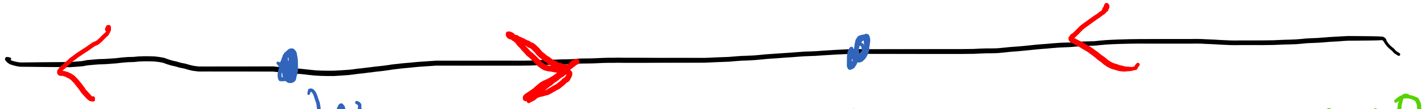
$\beta - \gamma < 0$

- $\frac{dI}{dt} = \left(\beta - \frac{\beta I}{N} - \gamma\right) I = (\beta - \gamma)\left[1 - \frac{\beta}{(\beta-\gamma)N} \cdot I\right] I = f(I)$

Eq: $\dot{I} = 0 \Rightarrow I = 0$ or $I = \frac{(\beta - \gamma)N}{\beta} < 0$

unstable

asymp. stable



$\underbrace{\frac{(\beta-\gamma)N}{\beta}}_{\text{irrelevant because negative}}$

0

$\underbrace{f(I) = (-)(+)(+) < 0}$

0 is a disease-free asymp. stable equilibrium

Case 3: $\beta > \gamma$

$\beta - \gamma > 0$

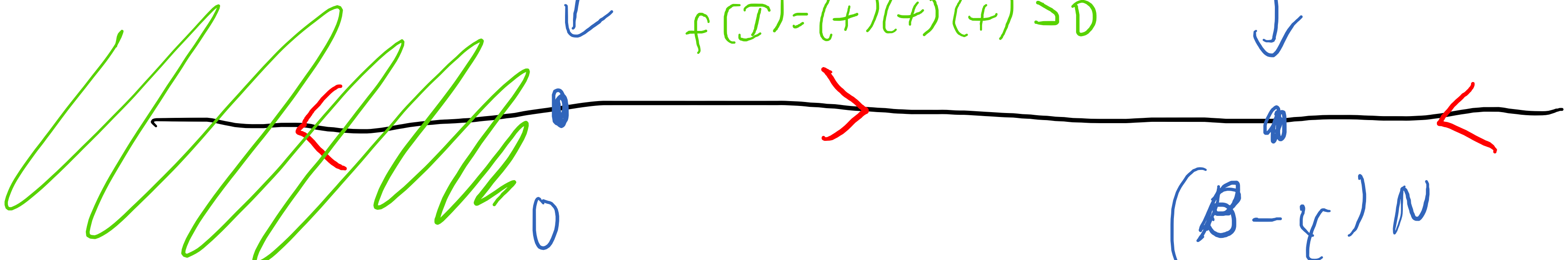
- $\frac{dI}{dt} = \left(\beta - \frac{\beta I}{N} - \gamma\right) I = (\beta - \gamma)\left[1 - \frac{\beta}{(\beta-\gamma)N} \cdot I\right] \cdot I = f(I)$

<u>Eq.</u> $\dot{I} = 0 \Rightarrow I = 0$ or $I = \frac{(\beta - \gamma)}{\beta} \cdot N > 0$

unstable

asymp. stable

$f(I) = (+)(+)(+) > 0$



$\frac{(\beta-\gamma)N}{\beta}$

Endemic asymp. stable equilibrium of $\frac{(\beta-\gamma)N}{\beta}$ infected

Basic reproduction number

- Case 1: $\beta = \gamma$. Disease dies out.
- Case 2: $\beta < \gamma$. Disease dies out.
- Case 3: $\beta > \gamma$. Disease persists.
- In the SIS model, we call the ratio $R_0 = \frac{\beta}{\gamma}$ the *basic reproduction number* of the system because it is the number of secondary infections β caused during the infectious period $\frac{1}{\gamma}$.
- When $R_0 > 1$, disease persists.
- When $R_0 \leq 1$, disease dies out.